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INDEX, NULLITY AND FLUX OF *n*-NOIDS

Dedicated to Professor Atsushi Kasue on his sixtieth birthday

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Abstract

In this paper, we give a criterion for 4-noids to have nullity greater than 3 and its applications. We also compute the indices and the nullities of some families of $\mathbf{Z}_N$-invariant n-noids, and analyze the correspondence between nullity and a flux map.

1. Introduction

Let M be a Riemann surface, and $X\colon M \to \mathbf{R}^3$ a complete conformal minimal immersion. The index of X is the supremum of the numbers of negative eigenvalues of the Jacobi operator $-\Delta - |dG|^2$ on relatively compact domains of M, where Δ is the Laplacian with respect to the metric $ds^2 = X^* ds^2_{\mathbf{R}^3}$ on M induced by X, and $G\colon M \to \mathbf{S}^2$ is the Gauss map of X. Fischer–Corbrie [5] and Gulliver-Lawson [6, 7] proved that X has a finite index if and only if it has finite total curvature, and Osserman [22] proved that if X has finite total curvature, then M is conformally equivalent with a compact Riemann surface $\bar{M}$ punctured by a finite number of points, and its Weierstrass data (g, η) extends meromorphically on $\bar{M}$.

If X has finite total curvature, then its index depends only on the extended Gauss map $G = \Pi^{-1} \circ g\colon \bar{M} \to \mathbf{S}^2 \subset \mathbf{R}^3$, where we denote the stereographic projection from the north pole by Π. Indeed, the index coincides with the number of negative eigenvalues of the operator $-\Delta^* - 2$, where Δ^* is the Laplacian with respect to the metric $G^* ds^2_{\mathbf{S}^2}$ on $\bar{M}$ induced by G. Hence we denote the index of X by both $\mathrm{Ind}(X)$ and $\mathrm{Ind}(g)$.

On the other hand, the nullity of X is defined as the dimension of the space of bounded Jacobi functions, elements of the kernel of the Jacobi operator. It also depends only on G since it coincides with the number of zero eigenvalues of $-\Delta^* - 2$. Hence we denote the nullity of X by both $\mathrm{Nul}(X)$ and $\mathrm{Nul}(g)$ in the same way as index.

Since there exists a 3-dimensional isometry group of parallel translations in $\mathbf{R}^3$, $\mathrm{Nul}(X) \geq 3$ holds for any X. The following fact is very significant since it characterizes nullity completely in a sense.

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Theorem 1.1 (Ejiri–Kotani [4], Montiel–Ros [21]). *If X has finite total curvature, then* $\mathrm{Nul}(X) > 3$ *holds if and only if its Gauss map is realized also as the Gauss map of some flat-ended non-branched or branched minimal surface.*

Other than this result, it is also known that if X has finite total curvature, and if all of its ends are embedded ends and parallel with each other, then $\mathrm{Nul}(X) > 3$ holds (cf. [16]). On the other hand, we see, by combining Nayatani's example in [20, §4] and basic facts, that there exists a family of X's such that $\mathrm{Nul}(X) > 3$, each of which has $N + 1$ catenoidal ends arranged on the positions of the vertices of a regular N-gonal pyramid (see Example 3.3).

By Theorem 1.1, each of these X's also has the same Gauss map as that of some flat-ended minimal surface. However, the reasons for nontrivial nullity seem to be different between flat or parallel ones and pyramidal ones, since the former have natural deformations which induce nontrivial bounded Jacobi functions, that is homotheties or rotations (or deformations to their associated family or López–Ros deformations if the genus of $\bar{M}$ is zero), but such deformations for the latter are not so trivial. What happens in the latter case? Which kind of X has the same Gauss map as a flat-ended surface in general? In particular, is some symmetry necessary?

Since the eigenvalues depend continuously on any parameter of deformations of X, index is lower semicontinuous, and nullity is upper semicontinuous with respect to the parameter. Therefore, determining the index and the nullity of some sampling point makes a significant role. For instance, Nayatani [18] showed that $\mathrm{Ind}(X) = 2(n-1) - 1 = 2n - 3$ and $\mathrm{Nul}(X) = 3$ hold for Jorge-Meeks' surface with n ends ($n \geq 3$). Since the moduli space of maps which are realized as the Gauss maps of some flat-ended minimal surface has codimension greater than 1 as a subset of the space of meromorphic maps of the common degree, Ejiri–Kotani [4] showed that if the genus of $\bar{M}$ is zero, that is $\bar{M} = \mathbf{S}^2 = \hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, then $\mathrm{Ind}(X) = 2d - 1$ and $\mathrm{Nul}(X) = 3$ hold for a generic X such that $\deg g = d$. In particular, since there is no flat-ended minimal surface with $\deg g = 2$, $\mathrm{Ind}(X) = 2 \cdot 2 - 1 = 3$ and $\mathrm{Nul}(X) = 3$ hold for any X such that $\deg g = 2$. On the other hand, since there are many flat-ended minimal surfaces with $d = \deg g \geq 3$, $\mathrm{Nul}(X) > 3$ (and $\mathrm{Ind}(X) < 2d - 1$ also) holds for some X such that $\deg g \geq 3$.

In this paper, we study index and nullity of n-noids, complete conformal minimal immersions with n embedded ends. In §§2–3 we summarize basic facts on n-noids and flat-ended minimal surfaces respectively, and in §§4–5, we give a criterion for 4-noids to have nullity greater than 3, and its applications. In §§6–7, we compute the indices and the nullities of some families of $\mathbf{Z}_N$-invariant n-noids. In §8, we discuss the correspondence between nullity and a flux map.

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2. Basic facts on *n*-noids and their flux

Let $X\colon M = \bar{M} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ be a complete conformal minimal immersion with finite total curvature. We use the Enneper–Weierstrass representation formula of the following type:

$$X(z) = \operatorname{Re} \int^z {}^t(1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta.$$

The *flux vector* of the end q_j of X is defined by the integral

$$\varphi_j := \int_{\gamma_j} \vec{n}\, ds,$$

where γ_j is a loop surrounding q_j from the left, $\vec{n}$ is a unit conormal vector field along γ_j such that $(\gamma_j{}', \vec{n})$ is positively oriented, and ds is the line element of $X(M)$. φ_j is independent of the choice of γ_j. By divergence formula, or residue theorem, it always holds that $\sum_{j=1}^n \varphi_j = \mathbf{0}$. We call this equality the *flux formula*.

It is known that, if the end q_j is an embedded end, then it is asymptotic to a catenoid or a plane. We call such an end a *catenoidal end* or *planar end* respectively. It is also known that the flux vector of any embedded end is parallel to its limit normal. Hence we can define the *weight* of the embedded end q_j by $w(q_j) := \varphi_j/(4\pi G(q_j))$, where G is the Gauss map of X as before. In another word, the weight is the ratio of the size of the asymptotic catenoid of the end to the standard catenoid. $w(q_j) = 0$ holds if and only if the end q_j is a planar end.

We call X an *n-noid* if all the ends $q_1, \dots, q_n$ are embedded ends. For an n-noid X, we can rewrite the flux formula by using the weights as follows:

$$\sum_{j=1}^n w(q_j)G(q_j) = \mathbf{0}.$$

We call a suit of unit vectors $v_1, \dots, v_n$ and real numbers $a_1, \dots, a_n$ satisfying $\sum_{j=1}^n a_j v_j = \mathbf{0}$ a *flux data*. We say an n-noid or a flux data is of *TYPE* III (resp. *TYPE* I, *TYPE* II) if the flux vectors span a 3- (resp. 1-, 2-) dimensional vector space. Umehara, Yamada and the first author [11, 12, 13] proved that, for generic flux data of TYPE III (or TYPE II with $n \le 8$), there exists an n-noid X of genus zero satisfying $G(q_j) = v_j$, $w(q_j) = a_j$ $(j = 1, \dots, n)$.

In general, if the genus of $\bar{M}$ is zero, that is $\bar{M} = \hat{\mathbf{C}}$, then the Weierstrass data (g, η) of an n-noid $X\colon M = \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ with $q_j \ne \infty$, $g(q_j) = p_j \ne \infty$, $w(q_j) = a_j$ $(j = 1, \dots, n)$ is of the following form:

$$g(z) = \frac{P(z)}{Q(z)}, \quad \eta = -Q(z)^2\, dz \tag{2.1}$$

with

$$P(z) = \sum_{j=1}^{n} \frac{p_j b_j}{z - q_j}, \quad Q(z) = \sum_{j=1}^{n} \frac{b_j}{z - q_j} \tag{2.2}$$

and

$$\begin{cases} \displaystyle\sum_{k=1;k\neq j}^{n} b_j b_k \frac{p_k - p_j}{q_k - q_j} = a_j \in \mathbf{R}, \\ \displaystyle\sum_{k=1;k\neq j}^{n} b_j b_k \frac{\overline{p_j} p_k + 1}{q_k - q_j} = 0, \end{cases} \quad (j = 1, \dots, n). \tag{2.3}$$

Hence, to find an n-noid with the prescribed flux data, we have only to solve (2.3) as an algebraic equation. More precisely, For any given p_j, a_j $(j = 1, \dots, n)$ satisfying the balancing condition

$$\sum_{j=1}^{n} a_j v_j = \sum_{j=1}^{n} a_j \, {}^t\!\left(\frac{2\,\mathrm{Re}\, p_j}{|p_j|^2 + 1}, \frac{2\,\mathrm{Im}\, p_j}{|p_j|^2 + 1}, \frac{|p_j|^2 - 1}{|p_j|^2 + 1} \right) = {}^t(0, 0, 0),$$

if q_j, b_j $(j = 1, \dots, n)$ satisfy the equation (2.3), and if $P(z)$ and $Q(z)$ have no common zero, then the Weierstrass data (g, η) given by (2.1) with (2.2) realizes an n-noid such that

$$\begin{cases} g(q_j) = p_j, \\ w(q_j) = a_j, \end{cases} \quad (j = 1, \dots, n).$$

We note here that it is useful to rewrite the second equalities in (2.3) as $A\mathbf{b} = \mathbf{0}$ with

$$A := \begin{pmatrix} 0 & \dfrac{\overline{p_1} p_2 + 1}{q_2 - q_1} & \cdots & \dfrac{\overline{p_1} p_n + 1}{q_n - q_1} \\ \dfrac{\overline{p_2} p_1 + 1}{q_1 - q_2} & 0 & \cdots & \dfrac{\overline{p_2} p_n + 1}{q_n - q_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\overline{p_n} p_1 + 1}{q_1 - q_n} & \dfrac{\overline{p_n} p_2 + 1}{q_2 - q_n} & \cdots & 0 \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If $q_1, \dots, q_n$ and $b_1, \dots, b_n$ realizes some n-noid, then it must hold that $\det A = 0$ and $\mathbf{0} \neq \mathbf{b} \in \mathrm{Ker}\, A$. In particular, in the case $n = 4$, $\mathrm{rank}\, A = 3$ (resp. 2) holds if the data is of TYPE III (resp. TYPE II) (cf. [11, Proposition 3.2]).

We also note here that we can define the *relative weights* of end-pairs (q_j, q_k) $(j, k = 1, \dots, n;\ j \neq k)$ by

$$w_{jk} := b_j b_k \frac{p_k - p_j}{q_k - q_j},$$

which is conformal invariants satisfying $w_{kj} = w_{jk}$ and $\sum_{k=1;\,k\neq j}^{n} w_{jk} = w(q_j)$ (cf. [10, 9]).

In general, we may assume that $q_j \neq \infty$, $p_j \neq \infty$ ($j = 1, \ldots, n$) without loss of generality. However, in some cases, it is more useful to assume that some p_j's and q_j's are ∞. In such case, we need to modify the equation (2.3) and (2.2) as follows:
(1) The case that $q_1 = p_1 = \infty$ and $q_j \neq \infty$, $p_j \neq \infty$ ($j = 2, \ldots, n$):

$$
\begin{cases}
\displaystyle\sum_{k=2}^{n} b_1 b_k = a_1, \\
\displaystyle b_j b_1 + \sum_{k=2;\,k\neq j}^{n} b_j b_k \frac{p_k - p_j}{q_k - q_j} = a_j, \\
\displaystyle\sum_{k=2}^{n} b_1 b_k(-p_k) = 0, \\
\displaystyle b_j b_1 \overline{p_j} + \sum_{k=2;k\neq j}^{n} b_j b_k \frac{\overline{p_j} p_k + 1}{q_k - q_j} = 0,
\end{cases}
\qquad (j = 2, \ldots, n), \tag{2.4}
$$

and

$$
P(z) = -b_1 + \sum_{j=2}^{n} \frac{p_j b_j}{z - q_j}, \quad Q(z) = \sum_{j=2}^{n} \frac{b_j}{z - q_j}. \tag{2.5}
$$

(2) The case that $q_1 = p_1 = p_2 = \infty$ and $q_j \neq \infty$ ($j = 2,\ldots,n$), $p_j \neq \infty$ ($j = 3,\ldots,n$):

$$
\begin{cases}
\displaystyle\sum_{k=3}^{n} b_1 b_k = a_1, \\
\displaystyle\sum_{k=3}^{n} c_2 b_k \frac{-1}{q_k - q_2} = a_2, \\
\displaystyle b_j b_1 + b_j c_2 \frac{1}{q_2 - q_j} + \sum_{k=3;k\neq j}^{n} b_j b_k \frac{p_k - p_j}{q_k - q_j} = a_j, \\
\displaystyle b_1 c_2(-1) + \sum_{k=3}^{n} b_1 b_k(-p_k) = 0, \\
\displaystyle c_2 b_1 + \sum_{k=3}^{n} c_2 b_k \frac{p_k}{q_k - q_2} = 0, \\
\displaystyle b_j b_1 \overline{p_j} + b_j c_2 \frac{\overline{p_j}}{q_2 - q_j} + \sum_{k=3;k\neq j}^{n} b_j b_k \frac{\overline{p_j} p_k + 1}{q_k - q_j} = 0,
\end{cases}
\qquad (j = 3, \ldots, n), \tag{2.6}
$$

and

$$(2.7)\qquad P(z) = -b_1 + \frac{c_2}{z-q_2} + \sum_{j=3}^{n} \frac{p_j b_j}{z-q_j}, \quad Q(z) = \sum_{j=3}^{n} \frac{b_j}{z-q_j}.$$

3. Basic facts on flat-ended minimal surfaces

In this section, we summarize basic facts on flat-ended minimal surfaces in the style suitable for our situation.

Let $X\colon M = \bar{M} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ be a complete conformal minimal branched (or non-branched) immersion with finite total curvature. We call the end q_j is a *flat end* if it is asymptotic to a plane. A flat end is not necessarily an embedded end. We say X is *flat-ended* if all of the ends $q_1, \dots, q_n$ are flat ends.

Since minimal surfaces with embedded flat ends, that is planar ends, have a correspondence with Willmore surfaces in $\mathbf{R}^3$, they were studied in early years. In particular, Bryant [2] proved many significant results on their moduli spaces. However, to determine the indices and the nullities of minimal surfaces by applying Theorem 1.1, we have to consider minimal surfaces with non-embedded flat ends also.

Here we call the end q_j of X is of *order* k if at least one of η and $g^2\eta$ has a pole at q_j and the maximum of the orders at q_j is k. For the well-definedness of X, k must be greater than 1. The end is an embedded end if and only if $k = 2$. On the other hand, the end q_j of order k is a flat end if and only if q_j is a zero of g' of order at least $k-1$ (see [4, Proposition 3.5]).

Now, let X be of genus zero, that is $\bar{M} = \hat{\mathbf{C}}$. We may assume that $q_j \neq \infty$ ($j = 1, \dots, n$) without loss of generality as before. If q_j is an end of order k_j, then $k_j \geq 2$ must hold for the well-definedness of X around q_j ($j = 1, \dots, n$). On the other hand, since ∞ is not an end of X, both η and $g^2\eta$ do not have a pole at ∞, that is, both η/dz and $g^2\eta/dz$ have a zero of order at least 2 at ∞. Hence, if $\deg g = d > 0$, then it must hold that

$$\begin{aligned} 2d &= \deg \frac{g^2\eta}{\eta} \\ &\leq \max\left\{\deg\left\{\prod_{j=1}^{n}(z-q_j)^{k_j}\frac{\eta}{dz}\right\}, \deg\left\{\prod_{j=1}^{n}(z-q_j)^{k_j}\frac{g^2\eta}{dz}\right\}\right\} \\ &\leq \sum_{j=1}^{n} k_j - 2. \end{aligned}$$

Now we see that

$$\sum_{j=1}^{n} k_j \geq \max\{2n, 2d+2\}.$$

Moreover, if $\sum_{j=1}^{n} k_j > 2d+2$, then X has $\sum_{j=1}^{n} k_j - 2d - 2$ branch points if counting their multiplicities.

On the other hand, if X is flat-ended, then it holds that

$$\sum_{j=1}^{n}(k_j - 1) \leq \#\{z \in \bar{M} \mid g'(z) = 0\} = 2d - 2.$$

Combining these facts, we have the following:

Lemma 3.1. *Let X be a flat-ended conformal minimal branched* (*or non-branched*) *immersion of genus zero. Suppose that each end q_j of X is of order k_j* ($j = 1, \dots, n$), *and that* $\deg g = d > 0$. *Then it holds that*

$$\max\{2n, 2d + 2\} \leq \sum_{j=1}^{n} k_j \leq n + 2d - 2.$$

In particular, it must hold that $d \geq 3$ and $4 \leq n \leq 2d - 2$.

For instance, in the case $d = 3$, we have

$$\max\{2n, 8\} \leq \sum_{j=1}^{n} k_j \leq n + 4$$

and $n = 4$. Hence the orders of the ends must satisfy the following:

$$8 \leq \sum_{j=1}^{4} k_j \leq 8, \quad \{k_j\} = \{2, 2, 2, 2\}.$$

In §4, we give a classification and a characterization of the surfaces in this class.

On the other hand, in the case $d = 4$, we have

$$\max\{2n, 10\} \leq \sum_{j=1}^{n} k_j \leq n + 8$$

and $n = 4, 5$ or 6. In this case, there are the following five possibilities:

$$n = 4, \quad 10 \leq \sum_{j=1}^{4} k_j \leq 10, \quad \{k_j\} = \{2, 2, 2, 4\} \text{ or } \{2, 2, 3, 3\},$$

$$n = 5, \quad 10 \leq \sum_{j=1}^{5} k_j \leq 11, \quad \{k_j\} = \{2, 2, 2, 2, 2\} \text{ or } \{2, 2, 2, 2, 3\} \text{ (1 branch point)},$$

$$n = 6, \quad 12 \leq \sum_{j=1}^{6} k_j \leq 12, \quad \{k_j\} = \{2, 2, 2, 2, 2, 2\} \text{ (2 branch points)},$$

where we counted the multiplicity of branch points as before. It is known that $\{k_j\} = \{2, 2, 2, 2, 2\}$ is not the case (cf. [2]; see Remark 4.2 for a short proof of this fact). However, to give some estimate for nullity, we must consider the remaining cases.

The following result has also to be recalled here. For later use, we describe the statement by means of a $GL(2, \mathbf{C})$-action, in place of the $SO(3, \mathbf{C})$-action Bryant considered.

Lemma 3.2. *Let (g, η) be the Weierstrass data of a flat-ended minimal surface of genus zero. Then*

$$\left(\frac{\alpha g + \beta}{\gamma g + \delta}, (\gamma g + \delta)^2 \eta\right)$$

is also the Weierstrass data of some flat-ended minimal surface for any $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ such that $\alpha\delta - \beta\gamma \neq 0$.

Proof. By the assumption, all the residues of ${}^t((1 - g^2)\eta, \sqrt{-1}(1 + g^2)\eta, 2g\eta)$ vanish. Hence those of ${}^t(\{(\gamma g + \delta)^2 - (\alpha g + \beta)^2\}\eta, \sqrt{-1}\{(\gamma g + \delta)^2 + (\alpha g + \beta)^2\}\eta, 2(\alpha g + \beta)(\gamma g + \delta)\eta)$ also, because

$$\begin{aligned} &\begin{pmatrix} \{(\gamma g + \delta)^2 - (\alpha g + \beta)^2\}\eta \\ \sqrt{-1}\{(\gamma g + \delta)^2 + (\alpha g + \beta)^2\}\eta \\ 2(\alpha g + \beta)(\gamma g + \delta)\eta \end{pmatrix} \\ &= \begin{pmatrix} \alpha^2 - \beta^2 - \gamma^2 + \delta^2 & \sqrt{-1}(\alpha^2 + \beta^2 - \gamma^2 - \delta^2) & 2(-\alpha\beta + \gamma\delta) \\ \sqrt{-1}(-\alpha^2 + \beta^2 - \gamma^2 + \delta^2) & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 2\sqrt{-1}(\alpha\beta + \gamma\delta) \\ 2(-\alpha\gamma + \beta\delta) & 2\sqrt{-1}(-\alpha\gamma - \beta\delta) & 2(\alpha\delta + \beta\gamma) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} (1 - g^2)\eta \\ \sqrt{-1}(1 + g^2)\eta \\ 2g\eta \end{pmatrix}. \end{aligned}$$

Since this transformation is linear, the property that all the ends are flat is preserved (cf. [4, Proposition 3.1]). $\square$

By this lemma, we see that two rational functions g_1 and g_2 have the same index and nullity with each other if there exist Möbius transformations φ and F of $\hat{\mathbf{C}}$ satisfying $g_1 \circ \varphi = F \circ g_2$. In this paper, we say that these two functions g_1 and g_2 are *equivalent* with each other.

EXAMPLE 3.3 ($(N + 1)$-noids with pyramidal flux). Nayatani [20] showed that $\mathrm{Ind}(g_{N\,\&\,M}) = 2d - 2 = 2(N + M) - 2$ and $\mathrm{Nul}(g_{N\,\&\,M}) = 5$ hold for the map $g_{N\,\&\,M}(z) := z^N + z^{-M}$ ($N, M \in \mathbf{N}$, $N + M \geq 3$). Let N be an integer such that $N \geq 3$, and set $\zeta_N := e^{2\pi\sqrt{-1}/N}$. For the data

j	$1, \ldots, N$	$N+1$
p_j	$p\zeta_N{}^{j-1}$	∞
a_j	a	a'

with $p \in \mathbf{R} \setminus \{0, \pm 1\}$, $a \in \mathbf{R} \setminus \{0\}$ and $a' = Na(1-p^2)/(1+p^2)$, by solving the equation (2.4), we get the following Weierstrass data (cf. [8, Example 3.3]):

$$g_{\mathrm{pyr}}(z) = \frac{(N-1)(p^2-1)z^N + p^N\{(N+1)p^2 + (N-1)\}}{2Np^2z^{N-1}},$$
$$\eta_{\mathrm{pyr}} = -\frac{a}{2(N-1)p^2(p^2+1)}\left(\frac{2Np^2z^{N-1}}{z^N - p^N}\right)^2 dz.$$

This data realizes an $(N+1)$-noid whose flux vectors are arranged on the positions of the vertices of a regular N-gonal pyramid. Since

$$\frac{2Np^2}{(N-1)(p^2-1)\cdot\epsilon}g_{\mathrm{pyr}}(z) = \left(\frac{\epsilon}{z}\right)^{N-1} + \left(\frac{\epsilon}{z}\right)^{-1}$$

holds for

$$\epsilon = \left\{\frac{(N+1)p^2 + (N-1)}{(N-1)(p^2-1)}\right\}^{1/N} \cdot p,$$

g_{pyr} is equivalent with $z^{N-1} + z^{-1}$, that is a special case of $g_{N\,\&\,M}$, and hence $\mathrm{Ind}(g_{\mathrm{pyr}}) = 2d - 2 = 2N - 2 = 2(N+1) - 4$ and $\mathrm{Nul}(g_{\mathrm{pyr}}) = 5$ hold.

4. A criterion in the case $\deg g = 3$

In this section, we give a criterion for the rational functions of degree 3 to be the Gauss map of some flat-ended minimal surface.

As we have already seen in §3, for any flat-ended minimal surface such that $\deg g = 3$, each of its ends must be an embedded flat end, namely the surface is a flat-ended 4-noid. The structure of the space of flat-ended n-noids was already studied by Bryant [2] (see also Kusner–Schmidt [15]), and we can compute the index and the nullity of any flat-ended 4-noid by applying Nayatani's estimate for $g_{N\,\&\,M}$ with $(N, M) = (2, 1)$. First, we summarlize these facts in the style suitable for our consideration.

In the case of flat-ended n-noids, that is the case $a_j = 0$ $(j = 1, \ldots, n)$, the algebraic equation (2.3) is equivalent with the following equation:

$$\text{(4.1)} \qquad \begin{cases} \displaystyle\sum_{k=1;k\neq j}^{n} b_k \frac{1}{q_k - q_j} = 0, \\ \displaystyle\sum_{k=1;k\neq j}^{n} p_k b_k \frac{1}{q_k - q_j} = 0, \end{cases} \qquad (j = 1, \ldots, n).$$

Hence, to classify all of the flat-ended 4-noids, we have only to solve (4.1) with $n = 4$ completely as an algebraic equation with respect to q_j and b_j ($j = 1, \dots, 4$). Note here that the equation (4.1) is rewritten as $A_0\mathbf{b} = A_0\mathbf{c} = \mathbf{0}$ with

$$A_0 := \begin{pmatrix} 0 & \dfrac{1}{q_2 - q_1} & \cdots & \dfrac{1}{q_n - q_1} \\ \dfrac{1}{q_1 - q_2} & 0 & \cdots & \dfrac{1}{q_n - q_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{q_1 - q_n} & \dfrac{1}{q_2 - q_n} & \cdots & 0 \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{c} := \begin{pmatrix} p_1 b_1 \\ p_2 b_2 \\ \vdots \\ p_n b_n \end{pmatrix}.$$

One of the most typical examples is given by the data

j	1	2	3	4
p_j	p	$-p$	$p^{-1}\sqrt{-1}$	$-p^{-1}\sqrt{-1}$

with $p := (\sqrt{6} + \sqrt{2})/2$. By solving (4.1), we get a family of solutions

j	1	2	3	4
q_j	$-p^{-1}$	p^{-1}	$-p\sqrt{-1}$	$p\sqrt{-1}$
b_j	$-p^{-1}\sqrt{t}$	$-p^{-1}\sqrt{t}$	$p\sqrt{t}$	$p\sqrt{t}$

where $t \in \mathbf{C} \setminus \{0\}$ is a parameter of homothety. The Weierstrass data of the flat-ended 4-noids given by these solutions are as follows:

$$\begin{aligned} g_{\mathrm{tet}}(z) &:= \frac{\sqrt{3}z^2 + 1}{z(z^2 - \sqrt{3})}, \\ \eta &:= -t\left\{\frac{2\sqrt{2}z(z^2 - \sqrt{3})}{(z^2 - p^{-2})(z^2 + p^2)}\right\}^2 dz = -8t\left\{\frac{z(z^2 - \sqrt{3})}{z^4 + 2\sqrt{3}z^2 - 1}\right\}^2 dz. \end{aligned}$$

In §6, we will analyze a family of functions which includes g_{tet} as a special case. By Lemma 3.2, we see that the Weierstrass data

$$\left(\frac{\alpha g_{\mathrm{tet}} + \beta}{\gamma g_{\mathrm{tet}} + \delta}, (\gamma g_{\mathrm{tet}} + \delta)^2 \eta\right) = \left(\frac{\alpha P_{\mathrm{tet}} + \beta Q_{\mathrm{tet}}}{\gamma P_{\mathrm{tet}} + \delta Q_{\mathrm{tet}}}, -(\gamma P_{\mathrm{tet}} + \delta Q_{\mathrm{tet}})^2\, dz\right) \tag{4.2}$$

also realizes a flat-ended 4-noid for any $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ such that $\alpha\delta - \beta\gamma \neq 0$, where we set

$$P_{\mathrm{tet}}(z) := \frac{2\sqrt{2}(\sqrt{3}z^2 + 1)}{(z^2 - p^{-2})(z^2 + p^2)}, \quad Q_{\mathrm{tet}}(z) := \frac{2\sqrt{2}z(z^2 - \sqrt{3})}{(z^2 - p^{-2})(z^2 + p^2)}.$$

The following fact seems to be well known among the researchers of this field. Bryant [1, §5] pointed out it to classify Willmore immersions from $\mathbf{S}^2$ into $\mathbf{S}^3$ with Willmore energy 12π. It follows directly by (4.1).

Lemma 4.1. *If X is a flat-ended 4-noid, then its ends q_1, q_2, q_3, q_4 satisfy the condition that the cross ratio $q_{1234} := (q_1 - q_2)(q_3 - q_4)/(q_1 - q_3)(q_2 - q_4)$ coincides with $\zeta_6 = e^{\pi\sqrt{-1}/3}$ or $\overline{\zeta_6} = e^{-\pi\sqrt{-1}/3}$, that is, the ends can be arranged on the positions of the vertices of a regular tetrahedron.*

Proof. By the first equalities of (4.1), we have $\mathbf{0} \neq \mathbf{b} \in \operatorname{Ker} A_0$. Now, since $n = 4$, it holds that

$$0 = \det A_0 = \left\{ \frac{{q_{1234}}^2 - q_{1234} + 1}{q_{1234}(q_1 - q_4)(q_2 - q_3)} \right\}^2 .$$

This implies our assertion. $\square$

REMARK 4.2. By the second equalities of (4.1), we also have $\mathbf{0} \neq \mathbf{c} \in \operatorname{Ker} A_0$. Since $\deg g = n - 1$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent. Hence $\operatorname{rank} A_0$ must be smaller than or equal to $n - 2$. Therefore, also in the case $n = 5$, the cross ratio of each four of $\{q_1, \dots, q_5\}$ must be ζ_6 or $\overline{\zeta_6}$. However there are no arrangement of the ends which satisfies such a condition. Indeed, if $q_{1234} = \zeta_6$ and $q_{1235} = \overline{\zeta_6}$, then $q_{2543} = q_{1234}/q_{1235} = \zeta_3 \neq \zeta_6, \overline{\zeta_6}$. Hence there are no flat-ended 5-noids. This is an essence of the proof of the nonexistence result for $n = 5$ given by Bryant [2].

The following fact asserts that the inverse of the assertion of Lemma 4.1 is also true. It is a restatement of the classification by Bryant [1, §5] we have already mentioned before.

Lemma 4.3. *Let g be a rational function of $\deg g = 3$. If the cross ratio of the zeroes of g' coincides with ζ_6 or $\overline{\zeta_6}$, then g is equivalent with the Gauss map of one of the flat-ended 4-noids given by (4.2).*

Proof. By the assumption, $\hat{\mathbf{C}} \setminus \{z \mid g'(z) = 0\}$ is conformally equivalent with $\hat{\mathbf{C}} \setminus \{z \mid {g_{\mathrm{tet}}}'(z) = 0\} = \hat{\mathbf{C}} \setminus \{\pm p^{-1}, \pm p\sqrt{-1}\}$, where $p = (\sqrt{6} + \sqrt{2})/2$. Hence there exists a Möbius transformation φ such that $\{z \mid (g \circ \varphi)'(z) = 0\} = \{z \mid {g_{\mathrm{tet}}}'(z) = 0\}$. Set

$$g \circ \varphi(z) := \frac{\sum_{j=0}^{3} \alpha_j z^j}{\sum_{j=0}^{3} \beta_j z^j}.$$

Then it holds that

$$(g \circ \varphi)'(z) = \frac{(\alpha_3\beta_2 - \alpha_2\beta_3)(z^4 + 2\sqrt{3}z^2 - 1)}{\left(\sum_{j=0}^{3} \beta_j z^j\right)^2}, \quad \alpha_3\beta_2 - \alpha_2\beta_3 \neq 0.$$

Hence we have

$$\begin{cases} 2(\alpha_3\beta_1 - \alpha_1\beta_3) = 0, \\ 3(\alpha_3\beta_0 - \alpha_0\beta_3) + (\alpha_2\beta_1 - \alpha_1\beta_2) = 2\sqrt{3}(\alpha_3\beta_2 - \alpha_2\beta_3), \\ 2(\alpha_2\beta_0 - \alpha_0\beta_2) = 0, \\ \alpha_1\beta_0 - \alpha_0\beta_1 = -(\alpha_3\beta_2 - \alpha_2\beta_3), \end{cases}$$

from which it follows that $\alpha_1 = -\sqrt{3}\alpha_3$, $\alpha_2 = \sqrt{3}\alpha_0$, $\beta_1 = -\sqrt{3}\beta_3$ and $\beta_2 = \sqrt{3}\beta_0$, where we use the assumption $\deg g = 3$. Now, we see that

$$\begin{aligned} g \circ \varphi(z) &= \frac{\alpha_3 z^3 + \sqrt{3}\alpha_0 z^2 - \sqrt{3}\alpha_3 z + \alpha_0}{\beta_3 z^3 + \sqrt{3}\beta_0 z^2 - \sqrt{3}\beta_3 z + \beta_0} \\ &= \frac{\alpha_0(\sqrt{3}z^2 + 1) + \alpha_3 z(z^2 - \sqrt{3})}{\beta_0(\sqrt{3}z^2 + 1) + \beta_3 z(z^2 - \sqrt{3})} = \frac{\alpha_0 g_{\mathrm{tet}}(z) + \alpha_3}{\beta_0 g_{\mathrm{tet}}(z) + \beta_3} = F \circ g_{\mathrm{tet}}(z), \end{aligned}$$

where

$$F(w) = \frac{\alpha_0 w + \alpha_3}{\beta_0 w + \beta_3}$$

and

$$\alpha_0\beta_3 - \alpha_3\beta_0 = \frac{1}{\sqrt{3}}\alpha_2 \cdot \beta_3 - \alpha_3 \cdot \frac{1}{\sqrt{3}}\beta_2 = -\frac{1}{\sqrt{3}}(\alpha_3\beta_2 - \alpha_2\beta_3) \neq 0. \qquad \square$$

By combining Lemmas 4.1 and 4.3, we see that the Weierstrass data of any flat-ended 4-noid is given by (4.2) up to conformal coordinate transformations. In particular, all the elements have the common index and nullity. The function $z^2 + z^{-1}$, that is one of Nayatani's examples $g_{N\,\&\,M}(z)$ with $(N, M) = (2, 1)$, is also in this case. Indeed, if we choose Möbius transformations

$$\varphi(z) := -2^{1/6} \cdot \frac{z - p}{pz + 1}, \quad F(w) := \frac{3}{2^{1/6}} \cdot \frac{pw + 1}{w - p}$$

with $p = (\sqrt{6} + \sqrt{2})/2$, then we have $F \circ g_{\mathrm{tet}}(z) = g_{N\,\&\,M} \circ \varphi(z)$. Hence, for any flat-ended 4-noid, its index and nullity must be 4 and 5 respectively. Now, we get the following:

Lemma 4.4. *Let X be a conformal minimal immersion of genus zero such that* $\deg g = 3$. *If the cross ratio of the zeroes of g' coincides with ζ_6 or $\overline{\zeta_6}$, then* $\mathrm{Ind}(X) = 4$ *and* $\mathrm{Nul}(X) = 5$ *hold. Otherwise,* $\mathrm{Ind}(X) = 5$ *and* $\mathrm{Nul}(X) = 3$ *hold.*

Let us give a criterion for the assumption in Lemma 4.4, which we will use in §5. First we prepare a criterion for polynomials.

Lemma 4.5. *Set* $f(z) := \sum_{j=0}^{4} a_j z^j$ $(a_4 \neq 0)$. *Then* $\hat{\mathbf{C}} \setminus \{z \mid f(z) = 0\}$ *is conformally equivalent with* $\hat{\mathbf{C}} \setminus \{z \mid g_{\text{tet}}'(z) = 0\}$ *if and only if* $D_{\text{tet0}} := a_2{}^2 - 3a_3 a_1 + 12 a_0 a_4 = 0$ *holds.*

Proof. Let $\{z_1, z_2, z_3, z_4\}$ be the set of solutions of $f(z) = 0$. Then $\hat{\mathbf{C}} \setminus \{z_1, z_2, z_3, z_4\}$ is conformally equivalent with $\hat{\mathbf{C}} \setminus \{z \mid g_{\text{tet}}'(z) = 0\}$ if and only if its cross ratio $z_{1234} := (z_1 - z_2)(z_3 - z_4)/(z_1 - z_3)(z_2 - z_4)$ coincides with either ζ_6 or $\overline{\zeta_6}$, that is, $z_{1234}{}^2 - z_{1234} + 1 = 0$. This equality is equivalent with

$$\begin{aligned} 0 &= (z_1 - z_2)^2 (z_3 - z_4)^2 - (z_1 - z_2)(z_3 - z_4)(z_1 - z_3)(z_2 - z_4) + (z_1 - z_3)^2 (z_2 - z_4)^2 \\ &= \sum_{i<j} z_i{}^2 z_j{}^2 - \sum_{i<j, i \neq k, j \neq k} z_i z_j z_k{}^2 + 6 z_1 z_2 z_3 z_4 =: D_1. \end{aligned}$$

Denote the elementary symmetric expression of degree j by σ_j, and set $\sigma_{2,2} := \sum_{i<j} z_i{}^2 z_j{}^2$ and $\sigma_{1,1,2} := \sum_{i<j;\, k \neq i,j} z_i z_j z_k{}^2$. Then, since $D_1 = \sigma_{2,2} - \sigma_{1,1,2} + 6\sigma_4$, $\sigma_2{}^2 = \sigma_{2,2} + 2\sigma_{1,1,2} + 6\sigma_4$ and $\sigma_1 \sigma_3 = \sigma_{1,1,2} + 4\sigma_4$, we have

$$\begin{aligned} D_1 &= \sigma_{2,2} + 2\sigma_{1,1,2} + 6\sigma_4 - 3\sigma_{1,1,2} = \sigma_2{}^2 - 3(\sigma_1 \sigma_3 - 4\sigma_4) = \sigma_2{}^2 - 3\sigma_1 \sigma_3 + 12\sigma_4 \\ &= \left(\frac{a_2}{a_4}\right)^2 - 3\left(-\frac{a_3}{a_4}\right)\left(-\frac{a_1}{a_4}\right) + 12\frac{a_0}{a_4} = \frac{1}{a_4{}^2}(a_2{}^2 - 3a_3 a_1 + 12 a_0 a_4) = \frac{D_{\text{tet0}}}{a_4{}^2}. \quad \square \end{aligned}$$

As a corollary to this lemma, we have a criterion for rational functions.

Lemma 4.6. *Let* $g(z) = \alpha(z)/\beta(z)$ *be a rational function of* $\deg g = 3$. *Set* $\alpha(z) := \sum_{j=0}^{3} \alpha_j z^j$ *and* $\beta(z) := \sum_{j=0}^{3} \beta_j z^j$. *Then* $\hat{\mathbf{C}} \setminus \{z \mid g'(z) = 0\}$ *is conformally equivalent with* $\hat{\mathbf{C}} \setminus \{z \mid g_{\text{tet}}'(z) = 0\}$ *if and only if* $D_{\text{tet}} := 3\alpha_3 \beta_0 - \alpha_2 \beta_1 + \alpha_1 \beta_2 - 3\alpha_0 \beta_3 = 0$ *holds.*

Proof. By applying Lemma 4.5 to $f(z) = \alpha'(z)\beta(z) - \alpha(z)\beta'(z)$, we have

$$D_{\text{tet0}} = (3\alpha_3 \beta_0 - \alpha_2 \beta_1 + \alpha_1 \beta_2 - 3\alpha_0 \beta_3)^2 = D_{\text{tet}}{}^2. \qquad \square$$

By combining Lemmas 4.4 and 4.6, we get the following:

Theorem 4.7. *Let* $g(z) = \sum_{j=0}^{3} \alpha_j z^j / \sum_{j=0}^{3} \beta_j z^j$ *be a rational function of* $\deg g = 3$. *If* $D_{\text{tet}} = 3\alpha_3 \beta_0 - \alpha_2 \beta_1 + \alpha_1 \beta_2 - 3\alpha_0 \beta_3 = 0$, *then* $\operatorname{Ind}(g) = 4$ *and* $\operatorname{Nul}(g) = 5$ *hold. Otherwise*, $\operatorname{Ind}(g) = 5$ *and* $\operatorname{Nul}(g) = 3$ *hold.*

5. Index and nullity of 4-noids

In this section, we observe which kind of 4-noid has the same Gauss map as that of a flat-ended 4-noid by applying Theorem 4.7.

As we have already mentioned in introduction, any n-noid of TYPE I has non-trivial bounded Jacobi functions, and hence $\mathrm{Ind}(X) = 4$ and $\mathrm{Nul}(X) = 5$ hold for any 4-noid X of TYPE I. On the other hand, these equalities also hold for any 4-noid X of TYPE III whose flux vectors are arranged on the positions of the vertices of a regular trigonal pyramid.

It should be remarked here that each 4-noid in these two families is located at a special position in the space of 4-noids from the viewpoint of the equation $\det A = 0$. Indeed, for any flux data of TYPE I, $\det A = 0$ is automatically satisfied and suitable conformal classes cannot be decided only by $\det A = 0$. On the other hand, for any flux data of TYPE III, the number of suitable conformal classes is at most 4, since $\det A = 0$ is equivalent with a quartic equation on the cross ratio of the ends (cf. [11, §3]). However, for any data of pyramidal type as above, the number is 2, that is, pyramidal examples are given by double solutions of the equation $\det A = 0$.

Hence it seems that there is some correspondence between the equation $\det A = 0$ and nullity, and the similar phenomenon is also expected in the case of TYPE II. However the condition that the cross ratio of the ends of X is given by a double solution of $\det A = 0$ is not a sufficient condition for $\mathrm{Nul}(X) > 3$. Indeed, for any flux data of TYPE II, each 4-noid is given by a double solution of $\det A = 0$ by the reason we describe below. But, for instance, the nullity of Jorge-Meeks' 4-noid is 3.

Here we present a result similar to above in the case of quadruple solutions.

Theorem 5.1. *If a 4-noid X is of TYPE II, and if its conformal class is given by a unique quadruple solution of the equation $\det A = 0$ on the cross ratio of the ends for some given flux data, then $\mathrm{Ind}(X) = 4$ and $\mathrm{Nul}(X) = 5$ hold.*

Proof. Since $\deg g = 4 - 1 = 3$ holds for any 4-noid X: $M = \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3, q_4\} \to \mathbf{R}^3$, the limit normals p_1, p_2, p_3, p_4 must take at least two distinct values.

First, we consider the case that at least one of p_j's is different from the others. In this case, we may assume that p_1 is different from the others, and in particular $p_1 = \infty$ without loss of generality. Since we can also choose three q_j's freely, we assume here that $q_1 = \infty$, $q_2 = 0$ and $q_3 = 1$.

For the data and the assumption

j	1	2	3	4
p_j	∞	p_2	p_3	p_4
q_j	∞	0	1	q

with $p_2,\ p_3,\ p_4 \in \mathbf{R}$ and $q \in \hat{\mathbf{C}} \setminus \{\infty, 0, 1\}$, set

$$A := \begin{pmatrix} 0 & -p_2 & -p_3 & -p_4 \\ p_2 & 0 & p_2p_3+1 & \dfrac{p_2p_4+1}{q} \\ p_3 & -(p_3p_2+1) & 0 & \dfrac{p_3p_4+1}{q-1} \\ p_4 & \dfrac{p_4p_2+1}{-q} & \dfrac{p_4p_3+1}{1-q} & 0 \end{pmatrix}.$$

If the equation (2.4) has a solution, then it holds that $\det A = 0$ and $\mathbf{0} \neq \mathbf{b} = {}^t(b_1, b_2, b_3, b_4) \in \operatorname{Ker} A$. In particular, since A is an alternative matrix, the pfaffian $\operatorname{Pf} A$ of A, that is a homogeneous polynomial of components of A satisfying $\det A = (\operatorname{Pf} A)^2$, is also defined, and given by

$$\operatorname{Pf} A = \frac{-1}{q(q-1)}\{p_2(p_3p_4+1)q - p_3(p_2p_4+1)(q-1) + p_4(p_2p_3+1)q(q-1)\}.$$

Set

$$\begin{aligned} \operatorname{pfa}(q) &:= -q(q-1)\operatorname{Pf} A \\ &= p_4(p_2p_3+1)q^2 + (-p_2p_3p_4 + p_2 - p_3 - p_4)q + p_3(p_2p_4+1). \end{aligned}$$

Then its derivative $\operatorname{pfa}'(q)$ and discriminant D_{pfa} of $\operatorname{pfa}(q)$ as a polynomial of q are given respectively by

$$\begin{aligned} \operatorname{pfa}'(q) &= 2p_4(p_2p_3+1)q + (-p_2p_3p_4 + p_2 - p_3 - p_4), \\ D_{\mathrm{pfa}} &:= \operatorname{pfa}'(q)^2 - 4p_4(p_2p_3+1)\operatorname{pfa}(q) \\ &= -3{p_2}^2{p_3}^2{p_4}^2 - 2{p_2}^2p_3p_4 - 2p_2{p_3}^2p_4 - 2p_2p_3{p_4}^2 \\ &\quad + {p_2}^2 + {p_3}^2 + {p_4}^2 - 2p_2p_3 - 2p_2p_4 - 2p_3p_4. \end{aligned}$$

Now, for any $\mathbf{b} \in \operatorname{Ker} A \setminus \{\mathbf{0}\}$, the corresponding Weierstrass data (g, η) is given by (2.1) with (2.5). Set $\alpha(z) := z(z-1)(z-q)P(z)$ and $\beta(z) := z(z-1)(z-q)Q(z)$. Then we have

$$\begin{aligned} \alpha(z) &= -b_1z^3 + \{(q+1)b_1 + p_2b_2 + p_3b_3 + p_4b_4\}z^2 \\ &\quad + \{-qb_1 - (q+1)p_2b_2 - qp_3b_3 - p_4b_4\}z + qp_2b_2, \\ \beta(z) &= (b_2+b_3+b_4)z^2 + \{-(q+1)b_2 - qb_3 - b_4\}z + qb_2. \end{aligned}$$

Since $\operatorname{Ker} A$ is spanned by ${}^t(p_2p_3+1, p_3, -p_2, 0)$ and ${}^t(-(p_2p_4+1), -p_4q, 0, p_2q)$, $\mathbf{b}$ is given by

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} s(p_2p_3+1) - t(p_2p_4+1) \\ sp_3 - tp_4q \\ -sp_2 \\ tp_2q \end{pmatrix}$$

for some $(s, t) \in \mathbf{C}^2 \setminus \{(0, 0)\}$. By direct computation, we have

$$\begin{aligned} D_{\mathrm{tet}}(q) &= (b_1b_2 + b_1b_3)q^2 + \{-2b_1b_2 - (p_2 - p_4)b_2b_4 - (p_3 - p_4)b_3b_4\}q \\ &\quad + \{b_1b_2 + b_1b_4 - (p_2 - p_3)b_2b_3 + (p_3 - p_4)b_3b_4\} \\ &= \{-(p_2p_3 + 1)(p_2 - p_3)q^2 - 2p_3(p_2p_3 + 1)q + p_3(p_2{}^2 + 1)\}s^2 \\ &\quad + \{-p_4(p_2p_3 + 1)q^3 + (p_2 - p_3 + 2p_4 + 2p_2p_3p_4)q^2 \\ &\quad + (p_2 + 2p_3 - p_4 + 2p_2p_3p_4)q - p_3(p_2p_4 + 1)\}st \\ &\quad + \{p_4(p_2{}^2 + 1)q^3 - 2p_4(p_2p_4 + 1)q^2 - (p_2p_4 + 1)(p_2 - p_4)q\}t^2, \end{aligned}$$

and

$$2p_4(p_2p_3 + 1)^2 D_{\mathrm{tet}}(q) = D_2(q, s, t)\,\mathrm{pfa}'(q) + C_1(s, t)\,\mathrm{pfa}(q)\,\mathrm{pfa}'(q) + C_2(s, t)\,\mathrm{pfa}(q),$$

where we set

$$\begin{aligned} D_2(q, s, t) &:= -2(p_2p_3 + 1)^2\{(p_2 - p_3)q + p_3\}s^2 \\ &\quad + 4(p_2p_3 + 1)\{(p_2 - p_3)q + p_3(p_2p_4 + 1)\}st \\ &\quad + 2[\{-(p_2 - p_3)(p_2{}^2 + 1) + p_2p_4(p_2p_3 + 1)(p_2 - p_4)\}q \\ &\quad - p_3(p_2{}^2 + 1)(p_2p_4 + 1)]t^2, \\ C_1(s, t) &:= -(p_2p_3 + 1)st + (p_2{}^2 + 1)t^2, \\ C_2(s, t) &:= 2(p_2 - p_3)(p_2p_3 + 1)^2s^2 \\ &\quad + (p_2p_3 + 1)\{-3(p_2 - p_3) + p_4(p_2p_3 + 1)\}st \\ &\quad + (p_2{}^2 + 1)\{(p_2 - p_3) - p_4(p_2p_3 + 1)\}t^2. \end{aligned}$$

Now, if the data (g, η) realizes a well-defined 4-noid X, then $\mathrm{pfa}(q) = 0$ and hence $2p_4(p_2p_3+1)^2 D_{\mathrm{tet}}(q) = D_2(q,s,t)\mathrm{pfa}'(q)$ holds. Moreover, if X is given by a quadruple solution, then $D_{\mathrm{pfa}} = 0$ and hence $\mathrm{pfa}'(q) = 0$ holds. If $p_4 = 0$ or $p_2p_3 + 1 = 0$, then $\mathrm{pfa}'(q) = p_2 - p_3 = 0$ must hold. However, in the case $p_4 = 0$, it is already known that this is not the case (cf. [11, Theorem 4.5]), and in the case $p_2p_3 + 1 = 0$, this contradicts the assumption that $p_2, p_3 \in \mathbf{R}$. Hence $p_4 \neq 0$ and $p_2p_3 + 1 \neq 0$, and we get $D_{\mathrm{tet}}(q) = 0$.

Secondly, we consider the case that p_1 takes the same value with p_2 only. In this case, we may assume that $p_1 = p_2 = \infty$ without loss of generality. Since we can also choose three q_j's freely, we assume here that $q_1 = \infty$, $q_2 = 0$ and $q_3 = 1$.

For the data and the assumption

j	1	2	3	4
p_j	∞	∞	p_3	p_4
q_j	∞	0	1	q

with $p_3, p_4 \in \mathbf{R}$ and $q \in \hat{\mathbf{C}} \setminus \{\infty, 0, 1\}$, set

$$\check{A} := \begin{pmatrix} 0 & -1 & -p_3 & -p_4 \\ 1 & 0 & p_3 & \dfrac{p_4}{q} \\ p_3 & -p_3 & 0 & \dfrac{p_3p_4+1}{q-1} \\ p_4 & \dfrac{p_4}{-q} & \dfrac{p_4p_3+1}{1-q} & 0 \end{pmatrix}.$$

If the equation (2.6) has a solution, then it holds that $\det \check{A} = 0$ and $\mathbf{0} \neq \check{\mathbf{b}} := {}^t(b_1, c_2, b_3, b_4) \in \operatorname{Ker} \check{A}$. In particular, since $\check{A}$ is also an alternative matrix, the pfaffian $\operatorname{Pf} \check{A}$ of $\check{A}$ is also defined, and given by

$$\operatorname{Pf} \check{A} = \frac{-1}{q(q-1)}\{(p_3p_4+1)q - p_3p_4(q-1) + p_3p_4q(q-1)\}.$$

Set

$$\operatorname{pfa}(q) := -q(q-1) \operatorname{Pf} \check{A} = p_3p_4q^2 + (-p_3p_4+1)q + p_3p_4.$$

Then its derivative $\operatorname{pfa}'(q)$ and discriminant D_{pfa} of $\operatorname{pfa}(q)$ as a polynomial of q are given respectively by

$$\begin{aligned} \operatorname{pfa}'(q) &= 2p_3p_4q + (-p_3p_4+1), \\ D_{\mathrm{pfa}} &:= \operatorname{pfa}'(q)^2 - 4p_3p_4 \operatorname{pfa}(q) \\ &= -3{p_3}^2{p_4}^2 - 2p_3p_4 + 1 = -(3p_3p_4-1)(p_3p_4+1). \end{aligned}$$

Now, for any $\check{\mathbf{b}} \in \operatorname{Ker} \check{A} \setminus \{\mathbf{0}\}$, the corresponding Weierstrass data (g, η) is given by (2.1) with (2.7). Set $\alpha(z)$ and $\beta(z)$ as in the first case. Then we have

$$\begin{aligned} \alpha(z) = &-b_1z^3 + \{(q+1)b_1 + c_2 + p_3b_3 + p_4b_4\}z^2 \\ &+ \{-qb_1 - (q+1)c_2 - p_3qb_3 - p_4b_4\}z + qc_2, \\ \beta(z) = &(b_3+b_4)z^2 + (-qb_3 - b_4)z. \end{aligned}$$

Since Ker $\check{A}$ is spanned by ${}^t(p_3, p_3, -1, 0)$ and ${}^t(-p_4, -p_4q, 0, q)$, $\check{\mathbf{b}}$ is given by

$$\check{\mathbf{b}} = \begin{pmatrix} b_1 \\ c_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} sp_3 - tp_4 \\ sp_3 - tp_4q \\ -s \\ tq \end{pmatrix}$$

for some $(s, t) \in \mathbf{C}^2 \setminus \{(0, 0)\}$. By direct computation, we have

$$D_{\mathrm{tet}}(q) = (q^2 - 1)(-p_3s^2 + p_4qt^2),$$

and

$$2{p_3}^2 p_4 D_{\mathrm{tet}}(q) = D_2(q, s, t)\,\mathrm{pfa}'(q) + C_1(t)\,\mathrm{pfa}(q)\,\mathrm{pfa}'(q) + C_2(s, t)\,\mathrm{pfa}(q),$$

where we set

$$\begin{aligned} D_2(q, s, t) &:= 2p_3q(-p_3s^2 + p_4qt^2), \\ C_1(t) &:= -t^2, \\ C_2(s, t) &:= 2{p_3}^2s^2 - (p_3p_4 - 1)t^2. \end{aligned}$$

Now, if the data (g, η) realizes a well-defined 4-noid X, then $\mathrm{pfa}(q) = 0$ and hence $2{p_3}^2 p_4 D_{\mathrm{tet}}(q) = D_2(q, s, t)\,\mathrm{pfa}'(q)$ holds. Moreover, if X is given by a quadruple solution, then $D_{\mathrm{pfa}} = 0$ and hence $\mathrm{pfa}'(q) = 0$ holds. If $p_3 = 0$ or $p_4 = 0$, then $\mathrm{pfa}'(q) = 1$ must hold. This contradicts $\mathrm{pfa}'(q) = 0$. Hence $p_3 \neq 0$ and $p_4 \neq 0$, and we get $D_{\mathrm{tet}}(q) = 0$.

Now, in both cases, by applying Theorem 4.7, we get our assertion. □

It is clear from the proof of Theorem 5.1 that the sufficient condition $\mathrm{pfa}'(q) = 0$ is valid independent of the choice of the parameters s and t. On the other hand, we can see also by the proof of Theorem 5.1 that $D_2(q, s, t) = 0$ also implies $D_{\mathrm{tet}}(q) = 0$. Since this condition depends on the choice of s and t, it comes from another type of deformation.

Before concluding this section, we present a description of the condition $D_{\mathrm{tet}} = 0$ by means of relative weights.

Theorem 5.2. *Let X be a 4-noid of genus zero. Then* $\mathrm{Ind}(X) = 4$ *and* $\mathrm{Nul}(X) = 5$ *hold if and only if its relative weights and cross ratios satisfy the following condition*:

$$(w_{12} + w_{34}) + (w_{13} + w_{24}){q_{1324}}^2 + (w_{14} + w_{23}){q_{1423}}^2 = 0. \tag{5.1}$$

This condition holds if the relative weights satisfy the following condition:

$$
\begin{cases} w_{\sigma(1)\sigma(2)}w_{\sigma(3)\sigma(4)} \neq w_{\sigma(1)\sigma(3)}w_{\sigma(2)\sigma(4)} \quad (\forall \sigma \in S_4), \\ (w_{12}+w_{34})(w_{13}w_{24}-w_{14}w_{23})^2+(w_{13}+w_{24})(w_{14}w_{23}-w_{12}w_{34})^2 \\ \ +(w_{14}+w_{23})(w_{12}w_{34}-w_{13}w_{24})^2=0. \end{cases} \tag{5.2}
$$

Proof. We may assume that $q_j \neq \infty$ ($j=1,2,3,4$) without loss of generality. Under this assumption, the Weierstrass data of X is given by (2.1) and (2.2). Set $\alpha(z) := P(z)\prod_{j=1}^{4}(z-q_j)$ and $\beta(z) := Q(z)\prod_{j=1}^{4}(z-q_j)$. Then, by direct computation, we have

$$
\begin{aligned}
D_{\mathrm{tet}} = (w_{12}+w_{34})(q_1-q_2)^2(q_3-q_4)^2 + (w_{13}+w_{24})(q_1-q_3)^2(q_2-q_4)^2 \\
+ (w_{14}+w_{23})(q_1-q_4)^2(q_2-q_3)^2,
\end{aligned}
$$

from which the condition (5.1) follows.

Now, if the inequalities in (5.2) hold, then the cross ratios of the ends are given by the following:

$$
\begin{aligned}
q_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)} &= \frac{(q_{\sigma(1)}-q_{\sigma(2)})(q_{\sigma(3)}-q_{\sigma(4)})}{(q_{\sigma(1)}-q_{\sigma(3)})(q_{\sigma(2)}-q_{\sigma(4)})} \\
&= \frac{w_{\sigma(1)\sigma(4)}w_{\sigma(2)\sigma(3)}-w_{\sigma(1)\sigma(3)}w_{\sigma(2)\sigma(4)}}{w_{\sigma(1)\sigma(4)}w_{\sigma(2)\sigma(3)}-w_{\sigma(1)\sigma(2)}w_{\sigma(3)\sigma(4)}} \quad (\forall \sigma \in S_4).
\end{aligned}
$$

By applying these equalities to (5.1), we get the equality in (5.2). □

As for the second assertion of Theorem 5.2, the condition given by the inequalities in (5.2) is a generic condition. Indeed, it means that the cross ratios of the ends and the limit normals are different from each other, and these cross ratios coincide with each other only if the limit normals $p_1,\ p_2,\ p_3,\ p_4$ satisfies

$$
\det A|_{(q_1,q_2,q_3,q_4)=(p_1,p_2,p_3,p_4)} = \det\left(\frac{\overline{p_j}\,p_k+1}{p_k-p_j}\right)_{j,k=1,2,3,4} = 0.
$$

In the case that this equality holds, a 4-noid realizing the corresponding relative weights is not unique, and we cannot determine the index and the nullity of such a 4-noid only by its relative weights.

6. Flat-ended minimal surfaces of $\deg g \geq 4$

In this section, we want to determine the nullities and the indices of a family of rational functions of degree greater than or equal to 3, which includes g_{tet} and $g_{N\,\&\,M}$ as special cases.

Let $N, L \in \mathbf{N}$, $L \leq N-1$, and let $s_{11}, s_{12}, s_{21}, s_{22} \in \mathbf{C} \setminus \{0\}$, $s_{11}s_{22} - s_{12}s_{21} \neq 0$. Set

$$g(z) := \frac{s_{11}z^N + s_{12}}{z^L(s_{21}z^N + s_{22})}.$$

In the case that some of $s_{11}, s_{12}, s_{21}, s_{22}$ are equal to 0, g is equivalent with z^{N+L}, z^{N-L}, $z^{N+L} + z^L$ or $z^{N-L} + z^{-L}$. The first (resp. second) one is the Gauss map of Jorge-Meeks' $(N+L+1)$- (resp. $(N-L+1)$-) noid, and Nayatani [18] proved that $\mathrm{Ind}(g) = 2d - 1 = 2(N+L+1) - 3$ (resp. $2(N-L+1)-3$) and $\mathrm{Nul}(g) = 3$ hold. Also for the last one, Nayatani [20] proved that $\mathrm{Ind}(g) = 2d - 2 = 2(N+1) - 4$ and $\mathrm{Nul}(g) = 5$ hold under the assumption $2 \leq L$ or $L \leq N-2$ (see Example 3.3).

Here we assume that each of s_{11}, s_{12}, s_{21}, s_{22} is not equal to 0. In this case, g is equivalent with

$$g_s(z) := \frac{sz^N + 1}{z^L(z^N - s)} \tag{6.1}$$

for some $s \in \mathbf{C} \setminus \{0\}$ satisfying $s^2 + 1 \neq 0$. Indeed, it holds that

$$g(\epsilon\tilde{z}) = \frac{s_{12}}{s_{21}\epsilon^{N+L}} \cdot g_s(\tilde{z})$$

with $\epsilon := (-s_{12}s_{22}/s_{11}s_{21})^{1/(2N)}$ and $s := (-s_{11}s_{22}/s_{12}s_{21})^{1/2}$. In the case $(N, L) = (2, 1)$, $g_{\sqrt{3}}$ coincides with the function g_{tet} which we considered in §4, and it is also equivalent with $z^2 + z^{-1}$. However g_s is not equivalent with $g_{N\,\&\,M}(z) = z^N + z^{-M}$ for a general pair (N, L), since the orders of zeroes of dG_s and $dG_{N\,\&\,M}$ do not coincide with each other in general, where $G_s := \Pi^{-1} \circ g_s$ and $G_{N\,\&\,M} := \Pi^{-1} \circ g_{N\,\&\,M}$.

Indeed, the derivative of g_s is given by

$$\begin{aligned} {g_s}'(z) &= \frac{-z^{L-1}\{Nz^N(s^2+1) + L(z^N - s)(sz^N + 1)\}}{z^{2L}(z^N - s)^2} \\ &= \frac{-[L(sz^{2N} - s) + \{(N-L)s^2 + (N+L)\}z^N]}{z^{L+1}(z^N - s)^2} \\ &= \frac{-Ls(z^N - t)(z^N + t^{-1})}{z^{L+1}(z^N - s)^2}, \end{aligned}$$

where $t \in \mathbf{C} \setminus \{0\}$ is a solution of the quadratic equation $Ls(t^2 - 1) + \{(N-L)s^2 + (N+L)\}t = 0$. Since $s^2 + 1 \neq 0$, it holds that $t - s \neq 0$ and $st + 1 \neq 0$. Now, assume that $(N-L)^2s^2 + (N+L)^2 \neq 0$ additionally. Then the equation above does not have a double solution, that is, $t^2 + 1 \neq 0$. Hence $q_{1,j} := t^{1/N}{\zeta_N}^{j-1}$ and $q_{2,j} := t^{-1/N}{\zeta_{2N}}^{2j-1}$ $(j = 1, \ldots, N)$ are the solutions of the algebraic equation $Ls(z^{2N} - 1) + \{(N-L)s^2 + (N+L)\}z^N = 0$, and zeroes of dG_s of order 1. Moreover, 0 and ∞ are zeroes of dG_s of order $L - 1$. On the other hand, the derivative of $g_{N\,\&\,M}$ is given by $g'_{N\,\&\,M}(z) =$

$(Nz^{N+M}-M)/z^{M+1}$, and hence $(M/N)^{1/(N+M)}\zeta_{N+M}^{j-1}$ $(j=1,\dots,N+M)$ are zeroes of $dG_{N\,\&\,M}$ of order 1, and 0 (resp. ∞) is a zero of $dG_{N\,\&\,M}$ of order $M-1$ (resp. $N-1$).

Kusner [14] gave an example of flat-ended minimal surface whose Gauss map is given by g_s in the case that $N \geq 2$, $L = N-1$ and $s=\sqrt{2N-1}$.

In general, if g_s is the Gauss map of some flat-ended minimal surface, then each of the ends of the surface must be a zero of dG_s. Moreover, if (g_s, η) is the Weierstrass data of the surface, then each of $q_{1,j}$, $q_{2,j}$ $(j=1,\dots,N)$ (resp. 0, ∞) is not a pole or a pole of η, $g_s\eta$, ${g_s}^2\eta$ whose order is 2 (resp. at most L if $L\geq 2$). Hence η must be of the following form:

$$\eta = \frac{z^L(z^N-s)^2h(z)}{(z^N-t)^2(z^N+t^{-1})^2}\,dz,$$

where h is a polynomial of degree at most $2N-2$. Here $g_s\eta$ and ${g_s}^2\eta$ is given by

$$g_s\eta = \frac{(z^N-s)(sz^N+1)h(z)}{(z^N-t)^2(z^N+t^{-1})^2}\,dz,$$
$${g_s}^2\eta = \frac{(sz^N+1)^2h(z)}{z^L(z^N-t)^2(z^N+t^{-1})^2}\,dz.$$

Set

$$h(z) := \sum_{l=0}^{2N-2} h_l z^l.$$

It is clear that both η and $g_s\eta$ do not have a pole on $z=0$. On the other hand, ${g_s}^2\eta$ has the following Laurent expansion near $z=0$.

$$\frac{{g_s}^2\eta}{dz} = \sum_{l=0}^{L-1}\frac{h_l}{z^{L-l}} + \text{“holomorphic part”}.$$

If $z=0$ is a regular point or a well-defined flat end, then the residue of this form must be zero, that is, $h_{L-1}=0$.

By the coordinate transformation $\tilde z := z^{-1}$, the Weierstrass data (g_s,η) is rewritten as follows:

$$g_s(\tilde z) = -\frac{\tilde z^L(\tilde z^N+s)}{s\tilde z^N-1},\quad \eta = -\frac{(s\tilde z^N-1)^2\tilde h(\tilde z)}{\tilde z^L(\tilde z^N-t^{-1})^2(\tilde z^N+t)^2}\,d\tilde z,$$

where $\tilde h$ is a polynomial defined by

$$\tilde h(\tilde z) := \tilde z^{2N-2}h(\tilde z^{-1}) = \sum_{l=0}^{2N-2} h_{2N-2-l}\tilde z^l.$$

Here $g_s\eta$ and ${g_s}^2\eta$ is given by

$$g_s\eta = \frac{(s\tilde{z}^N - 1)(\tilde{z}^N + s)\tilde{h}(\tilde{z})}{(\tilde{z}^N - t^{-1})^2(\tilde{z}^N + t)^2}\, d\tilde{z},$$
$${g_s}^2\eta = -\frac{\tilde{z}^L(\tilde{z}^N + s)^2\tilde{h}(\tilde{z})}{(\tilde{z}^N - t^{-1})^2(\tilde{z}^N + t)^2}\, d\tilde{z}.$$

It is clear that both $g_s\eta$ and ${g_s}^2\eta$ do not have a pole on $\tilde{z} = 0$. On the other hand, η has the following Laurent expansion near $\tilde{z} = 0$.

$$\frac{\eta}{d\tilde{z}} = \sum_{l=0}^{L-1} \frac{-h_{2N-2-l}}{\tilde{z}^{L-l}} + \text{“holomorphic part”.}$$

If $\tilde{z} = 0$, that is $z = \infty$, is a regular point or a well-defined flat end, then the residue of this form must be zero, that is, $h_{2N-L-1} = 0$.

Now, let us calculate the residues of η, $g_s\eta$ and ${g_s}^2\eta$ at $z = q_{1,j}$ $(j = 1, \ldots, N)$. By direct computation, we have the following expansions:

$$z^L = \frac{t}{{q_{1,j}}^{N-L}}\left\{1 + \frac{L}{q_{1,j}}(z - q_{1,j}) + O((z - q_{1,j})^2)\right\},$$
$$\frac{1}{z^L} = \frac{{q_{1,j}}^{N-L}}{t}\left\{1 - \frac{L}{q_{1,j}}(z - q_{1,j}) + O((z - q_{1,j})^2)\right\},$$
$$z^N - s = (t - s)\left\{1 + \frac{Nt}{(t - s)q_{1,j}}(z - q_{1,j}) + O((z - q_{1,j})^2)\right\},$$
$$sz^N + 1 = (st + 1)\left\{1 + \frac{Nst}{(st + 1)q_{1,j}}(z - q_{1,j}) + O((z - q_{1,j})^2)\right\},$$
$$\frac{1}{z^N + t^{-1}} = \frac{1}{t + t^{-1}}\left\{1 - \frac{Nt}{(t + t^{-1})q_{1,j}}(z - q_{1,j}) + O((z - q_{1,j})^2)\right\},$$
$$\frac{z - q_{1,j}}{z^N - t} = \frac{q_{1,j}}{Nt}\left\{1 - \frac{N - 1}{2q_{1,j}}(z - q_{1,j}) + O((z - q_{1,j})^2)\right\},$$

from which it follows that

$$\frac{z^L(z^N - s)^2}{(z^N - t)^2(z^N + t^{-1})^2} = \frac{t(t - s)^2{q_{1,j}}^{-N+L+2}}{N^2(t^2 + 1)^2}\left\{\frac{1}{(z - q_{1,j})^2} + \frac{{q_{1,j}}^{-1}\beta_{1,0}}{z - q_{1,j}} + O(1)\right\},$$
$$\frac{(z^N - s)(sz^N + 1)}{(z^N - t)^2(z^N + t^{-1})^2} = \frac{(t - s)(st + 1){q_{1,j}}^2}{N^2(t^2 + 1)^2}\left\{\frac{1}{(z - q_{1,j})^2} + \frac{{q_{1,j}}^{-1}\beta_{1,1}}{z - q_{1,j}} + O(1)\right\},$$
$$\frac{(sz^N + 1)^2}{z^L(z^N - t)^2(z^N + t^{-1})^2} = \frac{(st + 1)^2{q_{1,j}}^{N-L+2}}{N^2t(t^2 + 1)^2}\left\{\frac{1}{(z - q_{1,j})^2} + \frac{{q_{1,j}}^{-1}\beta_{1,2}}{z - q_{1,j}} + O(1)\right\},$$

where we set

$$\beta_{1,0} := -(N-L-1) + \frac{2Nt(st+1)}{(t^2+1)(t-s)},$$
$$\beta_{1,2} := -(N+L-1) - \frac{2Nt(t-s)}{(t^2+1)(st+1)},$$
$$\beta_{1,1} := \frac{\beta_{1,0}+\beta_{1,2}}{2}.$$

By the definition of t, we have

$$\begin{aligned}\beta_{1,0}-\beta_{1,2} &= -(N-L-1)+(N+L-1)+\frac{2Nt(st+1)}{(t^2+1)(t-s)}+\frac{2Nt(t-s)}{(t^2+1)(st+1)}\\ &= \frac{2\{Nt(s^2+1)+L(t-s)(st+1)\}}{(t-s)(st+1)} = 0,\end{aligned}$$

namely $\beta_{1,0}=\beta_{1,1}=\beta_{1,2}$. Denote this value by β_1. Then we get the following residues:

$$\operatorname{Res}_{z=q_{1,j}}\eta = \frac{t(t-s)^2 {q_{1,j}}^{-N+L+2}}{N^2(t^2+1)^2}(h'(q_{1,j})+{q_{1,j}}^{-1}\beta_1 h(q_{1,j})),$$
$$\operatorname{Res}_{z=q_{1,j}} g_s\eta = \frac{(t-s)(st+1){q_{1,j}}^2}{N^2(t^2+1)^2}(h'(q_{1,j})+{q_{1,j}}^{-1}\beta_1 h(q_{1,j})),$$
$$\operatorname{Res}_{z=q_{1,j}} {g_s}^2\eta = \frac{(st+1)^2{q_{1,j}}^{N-L+2}}{N^2 t(t^2+1)^2}(h'(q_{1,j})+{q_{1,j}}^{-1}\beta_1 h(q_{1,j})).$$

Since

$$\begin{aligned}h'(z)+z^{-1}\beta_1 h(z) &= z^{-1}(\beta_1 h(z)+zh'(z))\\ &= z^{-1}\sum_{l=0;\, l\neq L-1,2N-L-1}^{2N-2}(\beta_1+l)h_l z^l,\end{aligned}$$

we get the following conditions for the end $q_{1,j}$ to be a well-defined flat end:

$$\begin{aligned}0 &= \sum_{l=0;l\neq L-1,2N-L-1}^{2N-2}(\beta_1+l)h_l {q_{1,j}}^l\\ &= \sum_{l=0;l\neq L-1,2N-L-1}^{2N-2}(\beta_1+l)h_l t^{l/N}{\zeta_N}^{l(j-1)}.\end{aligned} \tag{6.2}$$

For any integer m such that $0 \le m \le N-1$, it holds that

$$
\begin{aligned}
0 &= \sum_{j=1}^{N} {\zeta_N}^{-m(j-1)} \sum_{l=0; l\neq L-1, 2N-L-1}^{2N-2} (\beta_1 + l) h_l t^{l/N} {\zeta_N}^{l(j-1)} \\
&= \sum_{l=0; l\neq L-1, 2N-L-1}^{2N-2} (\beta_1 + l) t^{l/N} h_l \sum_{j=1}^{N} {\zeta_N}^{(l-m)(j-1)} \qquad (6.3) \\
&= \begin{cases} N t^{m/N} \{(\beta_1 + m) h_m + (\beta_1 + m + N) t h_{m+N}\} & (m \in \mathbf{Z}_{N,L}), \\ N t^{m/N} (\beta_1 + m) h_m & (m = N-L-1,\ N-1), \\ N t^{m/N} (\beta_1 + m + N) t h_{m+N} & (m = L-1), \end{cases}
\end{aligned}
$$

where we set $\mathbf{Z}_{N,L} := \{m \in \mathbf{Z} \mid 0 \le m \le N-2,\ m \neq L-1,\ N-L-1\}$, and we use the equality

$$
\sum_{j=1}^{N} {\zeta_N}^{(l-m)(j-1)} = \begin{cases} 0 & (l \not\equiv m \bmod N), \\ N & (l \equiv m \bmod N). \end{cases}
$$

The condition (6.3) is equivalent with the original condition (6.2) ($j = 1, \ldots, N$).

By replacing t by $-t^{-1}$, we can also show that the ends $q_{2,j}$ ($j = 1, \ldots, N$) are well-defined flat ends if and only if

$$
0 = \begin{cases} N t^{-m/N} {\zeta_{2N}}^m \{(\beta_2 + m) h_m - (\beta_2 + m + N) t^{-1} h_{m+N}\} & (m \in \mathbf{Z}_{N,L}), \\ N t^{-m/N} {\zeta_{2N}}^m (\beta_2 + m) h_m & (m = N-L-1,\ N-1), \\ N t^{-m/N} {\zeta_{2N}}^m (-1)(\beta_2 + m + N) t^{-1} h_{m+N} & (m = L-1) \end{cases} \qquad (6.4)
$$

holds for any integer m such that $0 \le m \le N-1$, where we set

$$
\begin{aligned}
\beta_2 &:= -(N-L-1) + \frac{2Nt(t-s)}{(t^2+1)(st+1)} \\
&= -(N+L-1) - \frac{2Nt(st+1)}{(t^2+1)(t-s)} = -(2N-2) - \beta_1.
\end{aligned}
$$

Combining the conditions (6.3) and (6.4) for the ends $q_{1,j}$ and $q_{2,j}$ ($j = 1, \ldots, n$), we get the following:

(1) For $m = N-L-1$ or $N-1$ (resp. $L-1$), we can choose $h_m \neq 0$ (resp. $h_{m+N} \neq 0$) if and only if $\beta_1 + m = \beta_2 + m = 0$ (resp. $\beta_1 + m + N = \beta_2 + m + N = 0$).

(2) For $m \in \mathbf{Z}_{N,L}$, we can choose $(h_m, h_{m+N}) \neq (0, 0)$ if and only if

$$
\begin{aligned}
0 &= -\begin{vmatrix} \beta_1 + m & (\beta_1 + m + N)t \\ \beta_2 + m & -(\beta_2 + m + N)t^{-1} \end{vmatrix} \\
&= (\beta_1 + m)(\beta_2 + m + N)t^{-1} + (\beta_1 + m + N)t(\beta_2 + m) \\
&= t^{-1}\{(\beta_1 + m)(\beta_2 + m + N) + (\beta_1 + m + N)(\beta_2 + m)t^2\}.
\end{aligned}
$$

Since $\beta_1+\beta_2=-(2N-2)$, (1) is the case if and only if $m=-\beta_1=-\beta_2=N-1$. In this case, by $-\beta_1=N-1$ and the definition of t, we have

$$\frac{2(st+1)}{t^2+1}=\frac{-L(t-s)}{Nt}=\frac{s^2+1}{st+1},$$

from which it follows that $(st+1)^2=(s-t)^2$. Hence we have $t=(s-1)/(s+1)$ or $-(s+1)/(s-1)$. Now, by using the definition of t again, we get $s^2=(N+L)/(N-L)$.

For consider the situation (2), set

$$D_{N\,\&\,L}:=\frac{(\beta_1+m)(\beta_2+m+N)+(\beta_1+m+N)(\beta_2+m)t^2}{t^2+1}.$$

Then, by direct computation, we have

$$\begin{aligned}D_{N\,\&\,L}&=m^2+(\beta_1+\beta_2+N)m+\beta_1\beta_2+\frac{N(\beta_1+\beta_2t^2)}{t^2+1}\\&=m^2-(N-2)m+\frac{\beta_1\beta_2(t^2+1)+N(\beta_1+\beta_2t^2)}{t^2+1}.\end{aligned}$$

Moreover, by using the equalities

$$\begin{aligned}\beta_1\beta_2&=(\beta_1+N-L-1)(\beta_2+N-L-1)-(N-L-1)(\beta_1+\beta_2)-(N-L-1)^2\\&=\frac{2Nt(st+1)}{(t^2+1)(t-s)}\cdot\frac{2Nt(t-s)}{(t^2+1)(st+1)}+(N-L-1)(2N-2)-(N-L-1)^2\\&=\frac{4N^2t^2}{(t^2+1)^2}+(N-L-1)(N+L-1),\\\beta_1+\beta_2t^2&=(\beta_1+N-L-1)+(\beta_2+N-L-1)t^2-(N-L-1)(t^2+1)\\&=\frac{2Nt\{(st+1)^2+(t-s)^2t^2\}}{(t^2+1)(t-s)(st+1)}-(N-L-1)(t^2+1),\end{aligned}$$

we have

$$\begin{aligned}D_{N\,\&\,L}&=m^2-(N-2)m+(N-L-1)(N+L-1)-N(N-L-1)\\&\quad+\frac{2N^2t\{2t(t-s)(st+1)+(st+1)^2+(t-s)^2t^2\}}{(t^2+1)^2(t-s)(st+1)}\\&=m^2-(N-2)m+(N-L-1)(L-1)+\frac{2N^2t}{(t-s)(st+1)}\\&=m^2-(N-2)m+(N-L-1)(L-1)-\frac{2NL}{s^2+1}.\end{aligned}$$

Set

$$S_{N,L}(m):=\frac{2NL}{m^2-(N-2)m+(N-L-1)(L-1)}-1.$$

Then, $D_{N\,\&\,L} = 0$ holds if and only if $s^2 = S_{N,L}(m)$ holds for some $m \in \mathbf{Z}_{N,L}$.

Note here that $S_{N,L}(N-1) = (N+L)/(N-L)$, and that

$$\begin{aligned} S_{N,L}(m) &= \frac{2NL}{\{m-(N-2)/2\}^2-(N-2L)^2/4} - 1 \\ &= -\frac{\{m-(N+L-1)\}\{m-(-L-1)\}}{\{m-(N-L-1)\}\{m-(L-1)\}}. \end{aligned}$$

The latter implies that $S_{N,L}(m_1) = S_{N,L}(m_2)$ holds if and only if $m_1 = m_2$ or $m_1+m_2 = N-2$. It is clear that the matrix

$$\begin{pmatrix} \beta_1+m & (\beta_1+m+N)t \\ \beta_2+m & -(\beta_2+m+N)t^{-1} \end{pmatrix}$$

cannot be the zero matrix. Now, we get the following fact on $\mathrm{Nul}(g_s)$:

Theorem 6.1. *Let g_s be the rational function given by* (6.1). *Then the following assertions hold for its nullity*:
(1) *If $N \geq 2$ and $s^2 = S_{N,L}(N-1) = (N+L)/(N-L) > 0$, then* $\mathrm{Nul}(g_s) = 5$ *holds.*
(2) *If $N \geq 4$ and $s^2 \in \{S_{N,L}(m) \mid m \in \mathbf{Z},\ (N-1)/2 \leq m \leq N-2,\ m \neq L-1, N-L-1\}$, then* $\mathrm{Nul}(g_s) = 7$ *holds.*
(3) *If N is even, $N \geq 4$, $L \neq N/2$, and $s^2 = S_{N,L}((N-2)/2) = -(N+2L)^2/(N-2L)^2 < 0$, then* $\mathrm{Nul}(g_s) = 5$ *holds.*
(4) $\mathrm{Nul}(g_s) = 3$ *holds for any other s such that $s^2 \notin \{-1, -(N+L)^2/(N-L)^2\}$.*

In particular, if $\mathrm{Nul}(g_s) > 3$, then $s \in \mathbf{R} \cup \sqrt{-1}\mathbf{R}$. If $m < \min\{L-1,\ N-L-1\}$ or $\max\{L-1,\ N-L-1\} < m$, then $s \in \mathbf{R}$, and if $\min\{L-1,\ N-L-1\} < m < \max\{L-1,\ N-L-1\}$, then $s \in \sqrt{-1}\mathbf{R}$.

Since the set of s such that $\mathrm{Nul}(g_s) = 3$ is connected and includes 0, and since $g_0(z) = z^{-L-N}$, it holds that $\mathrm{Ind}(g_s) = \mathrm{Ind}(g_0) = 2d-1 = 2(N+L)-1$ for such s.

In the case that $m = N-1$ and $s^2 = (N+L)/(N-L)$, each of the flat-ended minimal surfaces above has the same symmetry as that of Costa's or Hoffman-Meeks' surfaces. Hence we can compute their indices by applying the method in Nayatani [20, 19] (see also [17]).

Set

$$\tau(s) := \frac{(N-L)s^2+(N+L)}{2Ls}.$$

Then, for any s, t is given by $t = -\tau(s) \pm \sqrt{\tau(s)^2+1}$. In particular, for any $s \in \mathbf{R}$, it also holds that $t \in \mathbf{R}$. Here we choose $t = -\tau(s) + \sqrt{\tau(s)^2+1}$. If $s > 0$, then $\tau(s) > 0$ and $t > 0$. Moreover, since

$$(s+\tau(s))^2 - (\tau(s)^2+1) = \frac{N}{L}(s^2+1) > 0,$$

we have $s > t$.

Set $I_1(z) := \bar{z}$, $I_2(z) := \zeta_N \bar{z}$ and

$$\eta_m := \frac{z^{L+m}(z^N - s)^2}{(z^N - t)^2(z^N + t^{-1})^2}\, dz.$$

Then it holds that

$$g_s(I_1(z)) = \overline{g_s(z)}, \quad {I_1}^*\eta_m = \overline{\eta_m},$$
$$g_s(I_2(z)) = {\zeta_N}^{-L}\overline{g_s(z)}, \quad {I_2}^*\eta_m = {\zeta_N}^{L+m+1}\overline{\eta_m}.$$

Let X_{Neu} (resp. X_{Dir}) be the flat-ended $2N$-noid given by the Weierstrass data $(g, \eta) = (g_{\sqrt{(N+L)/(N-L)}}, h_{N-1}\eta_{N-1})$ with $h_{N-1} \in \mathbf{R} \setminus \{0\}$ (resp. $h_{N-1} \in \sqrt{-1}\mathbf{R} \setminus \{0\}$).

Recall here that, for any conformal minimal immersion $X(z) = {}^t(X_1(z), X_2(z), X_3(z))$ whose Weierstrass data is given by (g, η), the following assertions hold:

(1) $X(I_1(z)) = \pm{}^t(X_1(z), -X_2(z), X_3(z))$ holds up to parallel translations if and only if (g, η) satisfies

$$(6.5_\pm) \qquad g(I_1(z)) = \overline{g(z)}, \quad {I_1}^*\eta = \pm\overline{\eta}.$$

(2) $X(I_2(z)) = \pm{}^t(\cos(2L\pi/N)X_1(z) - \sin(2L\pi/N)X_2(z), -\sin(2L\pi/N)X_1(z) - \cos(2L\pi/N)X_2(z), X_3(z))$ holds up to parallel translations if and only if (g, η) satisfies

$$(6.6_\pm) \qquad g(I_2(z)) = {\zeta_N}^{-L}\overline{g(z)}, \quad {I_2}^*\eta = \pm{\zeta_N}^{L}\overline{\eta}.$$

Since the Weierstrass data of X_{Neu} satisfies both of the conditions (6.5_+) and (6.6_+), it is symmetric with respect to both x_1x_3-plane and the plane $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 + \sqrt{-1}x_2 \in {\zeta_{2N}}^L\mathbf{R}\}$ up to parallel translations. Since $G_{\sqrt{(N+L)/(N-L)}} = \Pi^{-1} \circ g_{\sqrt{(N+L)/(N-L)}}$ also have the same symmetry as X_{Neu}, if we denote it by G, then it holds that

$$\langle X_{\mathrm{Neu}}(I_1(z)), G(I_1(z))\rangle = \langle X_{\mathrm{Neu}}(I_2(z)), G(I_2(z))\rangle = \langle X_{\mathrm{Neu}}(z), G(z)\rangle.$$

Hence we see that $\langle X_{\mathrm{Neu}}, G\rangle$ is an eigenfunction of the Laplacian Δ^* with respect to the metric $G^*(ds^2_{\mathbf{S}^2})$ on $\bar{M} = \hat{\mathbf{C}}$ which satisfies the Neumann boundary condition as an eigenfunction on the fundamental closed domain of the symmetry

$$\overline{\Omega} := \{z \in \hat{\mathbf{C}} \mid 0 \le \arg z \le \pi/N\}.$$

On the other hand, since the Weierstrass data of X_{Dir} satisfies both of the conditions (6.5_-) and (6.6_-), it holds that

$$\langle X_{\mathrm{Dir}}(I_1(z)), G(I_1(z))\rangle = \langle X_{\mathrm{Dir}}(I_2(z)), G(I_2(z))\rangle = -\langle X_{\mathrm{Dir}}(z), G(z)\rangle.$$

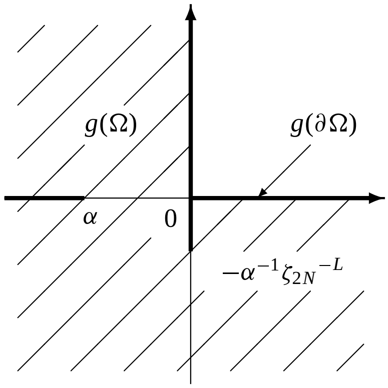

Fig. 6.1. The case $(N, L) = (2, 1)$.

Hence we see that $\langle X_{\mathrm{Dir}}, G\rangle$ is an eigenfunction of Δ^* which satisfies the Dirichlet boundary condition as an eigenfunction on $\overline{\Omega}$.

Now, the pushforwards of these functions by G can be regarded as eigenfunctions of the Laplacian with respect to the standard metric $ds^2_{\mathbf{S}^2}$ on the closed domain

$$\begin{aligned} G(\overline{\Omega}) &= \Pi^{-1} \circ g(\overline{\Omega}) \\ &= \left\{ \Pi^{-1}(w) \in \mathbf{S}^2 \;\middle|\; w \in \mathbf{C},\ -\frac{N+L}{N}\pi \le \arg w \le 0 \right\} \cup \{\Pi^{-1}(\infty)\}, \end{aligned}$$

which satisfy the Neumann or Dirichlet boundary condition if we regard

$$\begin{aligned} G(\partial\Omega) = \Pi^{-1} \circ g(\partial\Omega) = \{\Pi^{-1}(x) \in \mathbf{S}^2 \mid x \in \mathbf{R},\ x \le \alpha \text{ or } 0 \le x\} \\ \cup \{\Pi^{-1}(-x\zeta_{2N}{}^{-L}) \in \mathbf{S}^2 \mid x \in \mathbf{R},\ \alpha^{-1} \le x\} \cup \{\Pi^{-1}(\infty)\} \end{aligned}$$

as its boundary, where we set $\alpha := g_s(t^{1/N}) = (st+1)/\{t^{L/N}(t-s)\}$ (see Fig. 6.1).

Since $t < s$, $\alpha < 0$ holds. Moreover, since $g_s{}'(t^{1/N}) = 0$, we see that

$$\begin{aligned} \frac{d\alpha}{ds} &= \frac{d}{ds} g_s(t^{1/N}) = \frac{\partial g_s}{\partial s}(t^{1/N}) + g_s{}'(t^{1/N}) \frac{d}{ds}(t^{1/N}) \\ &= \frac{\partial}{\partial s}\left\{ \frac{sz^N + 1}{z^L(z^N - s)} \right\}\bigg|_{z=t^{1/N}} + 0 \cdot \frac{d}{ds}(t^{1/N}) \\ &= \frac{z^{2N}+1}{z^L(z^N-s)^2}\bigg|_{z=t^{1/N}} = \frac{t^2+1}{t^{L/N}(t-s)^2} > 0, \end{aligned}$$

that is, α is monotonically increasing with respect to s, and hence the boundary $G(\partial\Omega)$ is monotonically increasing and the domain $G(\Omega)$ is monotonically decreasing. In this situation, we can show that each Neumann (resp. Dirichlet) eigenvalue is monotonically non-increasing (resp. non-decreasing) with respect to s by the same way as [20, Lemmas 1 (b) and 6]. Since $\mathrm{Ind}(g_s) = 2d - 1 = 2(N + L) - 1$ and $\mathrm{Nul}(g_s) = 3$ hold for any s enough close to $\sqrt{(N+L)/(N-L)}$, and since $\mathrm{Nul}(g_{\sqrt{(N+L)/(N-L)}}) = 5$, it must hold that $\mathrm{Ind}(g_{\sqrt{(N+L)/(N-L)}}) = 2d - 2 = 2(N+L) - 2$.

Theorem 6.2. *Let g_s be the rational function given by* (6.1). *If $N \geq 2$ and $s^2 = S_{N,L}(N-1) = (N+L)/(N-L) > 0$, then $\mathrm{Ind}(g_s) = 2d - 2 = 2(N+L) - 2$ and $\mathrm{Nul}(g_s) = 5$ hold.*

7. Index and nullity of $\mathbf{Z}_N$-invariant n-noids

In this section, we give examples of n-noids with nontrivial nullity by applying the computations in §6.

EXAMPLE 7.1 (n-noids with parallel flux). As we have already mentioned in introduction, any n-noid with parallel ends has nontrivial nullity. Here we determine their indices for a typical case.

Let N be an integer such that $N \geq 2$. For the data

j	$1, \ldots, N$	$N+1$	$N+2$
p_j	0	0	∞
a_j	a	$-a(N-1)/2$	$a(N+1)/2$

with $a \in \mathbf{R} \setminus \{0\}$, by solving the equation (2.4), we get the following Weierstrass data:

$$g(z) = -\frac{1}{tf(z)}, \quad \eta = -taf(z)^2\,dz,$$

where f is a rational function given by

$$f(z) = \frac{(N+1)z^N + (N-1)}{z(z^N - 1)},$$

and $t \in \mathbf{R} \setminus \{0\}$ is a parameter of so-called López–Ros deformation. This data realizes a family of $\mathbf{Z}_N$-invariant $(N+2)$-noids of TYPE I for each $N \geq 2$ (cf. [11, Example 4.9]).

Now, since

$$f\left(\frac{z}{r}\right) = (N-1)r^{N+1}\frac{r^N z + 1}{z(z^N - r^N)}, \quad r = \left(\frac{N+1}{N-1}\right)^{1/2N},$$

the above g is equivalent with g_s with $s = \sqrt{(N+L)/(N-L)}$ and $(N, L) = (N, 1)$ in §6. Hence, by Theorem 6.2, we have $\mathrm{Ind}(g) = 2(N+1) - 2 = 2(N+2) - 4$ and $\mathrm{Nul}(g) = 5$.

EXAMPLE 7.2 ($2N$-noids with antiprismatic flux). Let N and M be integers such that $N \geq 2$, $1 \leq M \leq N-1$ and $(N, M) = 1$. For the data

j	$1, \ldots, N$	$N+1, \ldots, 2N$
p_j	$p\zeta_N{}^{M(j-1)}$	$p^{-1}\zeta_{2N}{}^{M(2j-1)}$
a_j	a	a

with $p \in \mathbf{R} \setminus \{0\}$ and $a \in \mathbf{R}$, by solving the equation (2.3), we get the following Weierstrass data:

$$g(z) = \frac{sz^N + 1}{z^{N-M}(z^N - s)}, \quad \eta = -t\frac{z^{2N-M-1}(z^N - s)^2}{(z^N - q^N)^2(z^N + q^{-N})^2}\,dz, \tag{7.1}$$

where

$$s = \frac{pq^{2N-M} - 1}{q^{N-M}(p + q^M)},$$

$q \in \mathbf{C} \setminus \{0\}$ satisfies

$$p^2 - 2\rho(q)p - 1 = 0, \quad \rho(q) = \frac{N}{N-M} \cdot \frac{q^{2N-M} - q^M}{q^{2N} + 1},$$

and $t \in \mathbf{R} \setminus \{0\}$ is a parameter of homothety chosen to satisfy

$$t = \frac{aN(p^2 - 1)(q^{2N} + 1)(p + q^M)^2}{(p^2 + 1)q^{2M}(p^2 q^{2N-2M} - 1)}$$

if $p^2 q^{2N-2M} - 1 \neq 0$.

In particular, in the case $q \in \mathbf{R}$, any $2N$-noid given by one of these data has the symmetry of a regular N-gonal antiprism, which has no branch point if $M = 1$ and $q \neq -1$.

Here we regard $q \in \mathbf{R} \setminus \{0\}$ as the parameter of deformation, and consider the case that

$$p = \rho(q) + \sqrt{\rho(q)^2 + 1} > 0.$$

In this case, by direct computation, we have

$$\frac{\partial p}{\partial q} = \frac{p}{\sqrt{\rho(q)^2 + 1}}\rho'(q),$$

$$\frac{N-M}{N}\rho'(q) = \frac{-q^{M-1}}{(q^{2N} + 1)^2}\varphi(q^2),$$

where we set

$$\varphi(t) := Mt^{2N-M} - (2N - M)t^N - (2N - M)t^{N-M} + M.$$

Hence $\partial p/\partial q = 0$ holds if and only if

$$\varphi(q^2) = M(q^{4N-2M} + 1) - (2N - M)q^{2N-2M}(q^{2M} + 1) = 0.$$

On the other hand, $s^2 = S_{N,N-M} = (2N - M)/M$ holds if and only if

$$0 = M(pq^{2N-M} - 1)^2 - (2N - M)\{q^{N-M}(p + q^M)\}^2 = \frac{\varphi(q^2)(p^2q^{2N-2M} - 1)}{q^{2N-2M} - 1}.$$

Namely, if $\partial p/\partial q = 0$, that is, q is a double solution of the equation $\rho(q) = (p^2 - 1)/(2p)$, then, by Theorem 6.2 again, we have $\mathrm{Ind}(g) = 2(2N - M) - 2$ and $\mathrm{Nul}(g) = 5$. $p^2q^{2N-2M} - 1 = 0$ with $q^{2N-2M} - 1 \neq 0$ is the case of flat-ended ones (cf. [14, Remarks 1 and 2] for $M = 1$).

In particular, in the case that $N = 2$, $M = 1$ and $q = p = (\sqrt{6} + \sqrt{2})/2$, the data (7.1) realizes a tetrahedrally symmetric 4-noid. Since $s^2 = 3 = S_{2,1}$, this is a special case of both the consideration above and Example 3.3, and hence its nullity is 5.

On the other hand, in the case that $N = 3$, $M = 1$ and $q = p = (\sqrt{6} + \sqrt{2})/2$, the data (7.1) realizes an octahedrally symmetric 6-noid. Since $s^2 = 25/2 \neq 5 = S_{3,2}$, by Theorem 6.1 and the fact for the indices of generic surfaces by Ejiri–Kotani [4] we introduced in the introduction, we get the following:

Theorem 7.3. *Let X be the octahedrally symmetric 6-noid as above. Then $\mathrm{Ind}(X) = 2 \cdot 6 - 3 = 9$ and $\mathrm{Nul}(X) = 3$ hold.*

Namely, symmetries of platonic solids do not always induce nontrivial bounded Jacobi functions.

8. Nullity and a flux map

In this section, we study the correspondence between nullity and a flux map.

In the case of n-noids of TYPE I and positive genus, Pérez–Ros [23] considered a map from the moduli space of such n-noids to the space of the weights and the heights of the ends, and defined the nondegeneracy of such n-noids mainly by the property that any bounded Jacobi function is a trivial one, that is, the nullity of the surface is $3 + 1 = 4$. By using these concepts, they analyzed the real analytic structure of the moduli space of such n-noids.

On the other hand, Umehara, Yamada and the first author [11, 12, 13] considered a flux map defined as a map from the parameter space of n-noids of genus zero with common limit normals to the space of the weights of the ends for each suit of limit normals, and proved that, for a generic flux data of TYPE III (or TYPE II with $n \leq 8$), there exists an n-noid of genus zero which realizes the given flux data, by showing that the rank of the Jacobian matrix of the flux map is maximal for generic parameters and limit normals.

For the case of TYPE II, genus zero and Alexandrov-embedded, Cosín–Ros [3] considered a flux map defined as a map from the moduli space of such n-noids to the space of flux polygons, that is the ordered flux vectors of the ends, and defined the nondenegeracy of such n-noids in the same way as in Pérez–Ros [23] with the condition that the nullity is 3. They proved that the flux map is a real analytic diffeomorphism on to the space of flux polygons each of which bounds an immersed disc in the plane.

From these points of view, it seems natural to expect, as the contraposition to some generalization of these results, that if an n-noid X is degenerate or a critical point of a flux map in some sense, then there exists a nontrivial bounded Jacobi function and $\mathrm{Nul}(X) > 3$. Indeed, we can show this for n-noids of arbitrary genus. Although its proof is given by a quite natural calculation, we show its detail here to observe the correspondence between Jacobi functions and flux precisely. Here we define a flux map in an essentially similar way as in [11, 12, 13]. Since we treat a situation different from that of [23], our consequence also takes a somewhat different form from that in [23].

Let U be an open subset of $\mathbf{C}$, and I an open interval in $\mathbf{R}$. Let $q(t)$ $(t \in I)$ be a smooth curve in U, and X: $(U \times I) \setminus \{(q(t), t) \mid t \in I\} \to \mathbf{R}^3$ a smooth 1-parameter family of conformal minimal immersions in the sense that $X(\,\cdot\,, t)$ is a conformal minimal immersion for each $t \in I$ and that both $\Pi^{-1} \circ g$ and $\Pi^{-1} \circ (\eta/dz)$ are smooth with respect to (z, t) as maps from $U \times I$ to $\mathbf{S}^2$, where (g, η) is the Weierstrass data of $X(\,\cdot\,, t)$. Assume that each of $X(\,\cdot\,, t)$ has a catenoidal or planar end at $q(t)$. Then the Taylor or Laurent expansions of (g, η) around $q = q(t)$ is of the following form:

$$g = p + \gamma(z - q) + (z - q)^2 g_2(z),$$
$$\eta = \left\{ \frac{B}{(z - q)^2} + \frac{b}{z - q} + f_0(z) \right\} dz,$$

where p, γ, B and b are smooth functions depending only on the parameter $t \in I$, and g_2 and f_0 are holomorphic functions on U both of which are smooth on I. By these expansions, it follows that

$$g\eta = \left\{ \frac{pB}{(z - q)^2} + \frac{pb + \gamma B}{z - q} + f_1(z) \right\} dz,$$
$$g^2\eta = \left\{ \frac{p^2 B}{(z - q)^2} + \frac{p^2 b + 2p\gamma B}{z - q} + f_2(z) \right\} dz$$

for some holomorphic functions f_1 and f_2, from which it also follows that

$$\phi_1 := \frac{(1 - g^2)\eta}{dz} = \frac{(1 - p^2)B}{(z - q)^2} + \frac{(1 - p^2)b - 2p\gamma B}{z - q} + (f_0(z) - f_2(z)),$$

$$\begin{aligned}\phi_2 &:= \frac{\sqrt{-1}(1+g^2)\eta}{dz}\\ &= \frac{\sqrt{-1}(1+p^2)B}{(z-q)^2} + \frac{\sqrt{-1}\{(1+p^2)b + 2p\gamma B\}}{z-q} + \sqrt{-1}(f_0(z)+f_2(z)),\\ \phi_3 &:= \frac{2g\eta}{dz} = \frac{2pB}{(z-q)^2} + \frac{2(pb+\gamma B)}{z-q} + 2f_1(z).\end{aligned}$$

Let (v, a) be the flux data of $X(\,\cdot\,, t)$. Then we have

$$\begin{aligned}&(1-p^2)b - 2p\gamma B = -2av_1 = \frac{-2a}{|p|^2+1}(p+\overline{p}),\\ &\sqrt{-1}\{(1+p^2)b + 2p\gamma B\} = -2av_2 = \frac{-2a}{|p|^2+1}(-\sqrt{-1})(p-\overline{p}),\\ &2(pb+\gamma B) = -2av_3 = \frac{-2a}{|p|^2+1}(|p|^2-1),\end{aligned}$$

and hence we get

$$a = \gamma B, \quad b = \frac{-2a}{|p|^2+1}\overline{p}.$$

By integrate the 1-forms above, we have

$$\begin{aligned}\Phi_1 &:= \int^z \phi_1\,dz = -\frac{(1-p^2)B}{z-q} - 2av_1\log(z-q) + (F_0 - F_2),\\ \Phi_2 &:= \int^z \phi_2\,dz = -\frac{\sqrt{-1}(1+p^2)B}{z-q} - 2av_2\log(z-q) + \sqrt{-1}(F_0+F_2),\\ \Phi_3 &:= \int^z \phi_3\,dz = -\frac{2pB}{z-q} - 2av_3\log(z-q) + 2F_1,\end{aligned}$$

where F_0, F_1 and F_2 are holomorphic functions on U each of which is smooth on I. Henceforth we denote the derivative with respect to the parameter t by $_t$ or $(\ )_t$. By differentiate Φ_1, Φ_2 and Φ_3 by t, we get

$$\begin{aligned}(\Phi_1)_t &= -\frac{q_t(1-p^2)B}{(z-q)^2} - \frac{-2pp_tB + (1-p^2)B_t - 2q_tav_1}{z-q}\\ &\quad - 2\{a_tv_1 + a(v_1)_t\}\log(z-q) + (F_0-F_2)_t,\\ (\Phi_2)_t &= -\frac{q_t\sqrt{-1}(1+p^2)B}{(z-q)^2} - \frac{\sqrt{-1}\{2pp_tB + (1+p^2)B_t\} - 2q_tav_2}{z-q}\\ &\quad - 2\{a_tv_2 + a(v_2)_t\}\log(z-q) + \sqrt{-1}(F_0+F_2)_t,\\ (\Phi_3)_t &= -\frac{q_t\cdot 2pB}{(z-q)^2} - \frac{2(p_tB + pB_t) - 2q_tav_3}{z-q} - 2\{a_tv_3 + a(v_3)_t\}\log(z-q) + 2(F_1)_t.\end{aligned}$$

On the other hand, since the Gauss map $G(\,\cdot\,,t)$ of $X(\,\cdot\,,t)$ is given by

$$G = {}^t(G_1, G_2, G_3) = \Pi^{-1} \circ g = \frac{1}{|g|^2+1}{}^t(g+\overline{g}, -\sqrt{-1}(g-\overline{g}), |g|^2-1),$$

by the expansion $g = p + \gamma(z-q) + (z-q)^2 g_2(z)$, we have

$$\begin{aligned}
G_1 &= \frac{1}{|g|^2+1}\{(p+\overline{p}) + \gamma(z-q) + \overline{\gamma}(\overline{z}-\overline{q}) + O(|z-q|^2)\},\\
G_2 &= \frac{1}{|g|^2+1}\{(-\sqrt{-1})(p-\overline{p}) + (-\sqrt{-1})\gamma(z-q) + \sqrt{-1}\overline{\gamma}(\overline{z}-\overline{q}) + O(|z-q|^2)\},\\
G_3 &= \frac{1}{|g|^2+1}\{(|p|^2-1) + \overline{p}\gamma(z-q) + p\overline{\gamma}(\overline{z}-\overline{q}) + O(|z-q|^2)\}.
\end{aligned}$$

Note here that $(p+\overline{p}, -\sqrt{-1}(p-\overline{p}), |p|^2-1) = (|p|^2+1)(v_1, v_2, v_3)$, ${v_1}^2+{v_2}^2+{v_3}^2 \equiv 1$ and $({v_1}^2+{v_2}^2+{v_3}^2)_t \equiv 0$. By direct computation, we have

$$\begin{aligned}
&(|g|^2+1)\langle \Phi_t, G\rangle = (|g|^2+1)\{(\Phi_1)_t G_1 + (\Phi_2)_t G_2 + (\Phi_3)_t G_3\}\\
&= \frac{1}{(z-q)^2}[-q_t B\{(1-p^2)(p+\overline{p}) + \sqrt{-1}(1+p^2)(-\sqrt{-1})(p-\overline{p}) + 2p(|p|^2-1)\}]\\
&\quad + \frac{1}{z-q}[-q_t\gamma B\{(1-p^2)\cdot 1 + \sqrt{-1}(1+p^2)(-\sqrt{-1}) + 2p\cdot\overline{p}\}\\
&\qquad\qquad - 2p_t B\{-p(p+\overline{p}) + \sqrt{-1}p(-\sqrt{-1})(p-\overline{p}) + (|p|^2-1)\}\\
&\qquad\qquad - B_t\{(1-p^2)(p+\overline{p}) + \sqrt{-1}(1+p^2)(-\sqrt{-1})(p-\overline{p}) + 2p(|p|^2-1)\}\\
&\qquad\qquad + 2q_t a\{v_1(p+\overline{p}) + v_2(-\sqrt{-1})(p-\overline{p}) + v_3(|p|^2-1)\}]\\
&\quad + \frac{\overline{z}-\overline{q}}{(z-q)^2}[-q_t\overline{\gamma}B\{(1-p^2)\cdot 1 + \sqrt{-1}(1+p^2)\cdot\sqrt{-1} + 2p\cdot p\}]\\
&\quad - 2\log(z-q)[a_t\{(p+\overline{p})v_1 + (-\sqrt{-1})(p-\overline{p})v_2 + (|p|^2-1)v_3\}\\
&\qquad\qquad\qquad + a\{(p+\overline{p})(v_1)_t + (-\sqrt{-1})(p-\overline{p})(v_2)_t + (|p|^2-1)(v_3)_t\}] + O(1)\\
&= \frac{1}{(z-q)^2}(-q_t B\cdot 0)\\
&\quad + \frac{1}{z-q}[-q_t\gamma B\cdot 2(|p|^2+1) + 2p_t B\cdot(|p|^2+1) - B_t\cdot 0\\
&\qquad\qquad + 2q_t a\{v_1\cdot v_1(|p|^2+1) + v_2\cdot v_2(|p|^2+1) + v_3\cdot v_3(|p|^2+1)\}]\\
&\quad + \frac{\overline{z}-\overline{q}}{(z-q)^2}(-q_t\overline{\gamma}B\cdot 0)\\
&\quad - 2\log(z-q)[a_t\{v_1(|p|^2+1)\cdot v_1 + v_2(|p|^2+1)\cdot v_2 + v_3(|p|^2+1)\cdot v_3\}\\
&\qquad\qquad\qquad + a\{v_1(|p|^2+1)\cdot(v_1)_t + v_2(|p|^2+1)\cdot(v_2)_t + v_3(|p|^2+1)\cdot(v_3)_t\}]\\
&\qquad\qquad\qquad + O(1)
\end{aligned}$$

$$= \frac{1}{z-q}\{-2q_t a(|p|^2+1) + 2p_t B(|p|^2+1) + 2q_t a({v_1}^2+{v_2}^2+{v_3}^2)(|p|^2+1)\}$$
$$- 2\log(z-q)\left\{a_t({v_1}^2+{v_2}^2+{v_3}^2)(|p|^2+1) + a\cdot\frac{1}{2}({v_1}^2+{v_2}^2+{v_3}^2)_t(|p|^2+1)\right\}$$
$$+ O(1)$$
$$= (|p|^2+1)\left\{\frac{2p_t B}{z-q} - 2a_t\log(z-q)\right\} + O(1).$$

Finally, we get

$$\langle \Phi_t, G\rangle = \frac{|p|^2+1}{|g|^2+1}\left\{\frac{2p_t B}{z-q} - 2a_t\log(z-q)\right\} + O(1). \tag{8.1}$$

Hence, if $p_t = 0$ and $a_t = 0$, then $\langle X_t, G\rangle = \mathrm{Re}\langle \Phi_t, G\rangle$ is bounded near q.

For later use, we also give here an estimate for $(X(\,\cdot\,,t))^*(X_t)$ in the special case that $\langle X_t, G\rangle = 0$ holds for some $t \in I$, that is, X_t is a tangent vector field of the image of $X(\,\cdot\,,t)$. If, for instance, $v_3 = G_3(q) \neq 0$, that is $|p| = |g(q)| \neq 1$, then it holds around q that

$$(X(\cdot,t))^*(X_t) = \frac{1}{(X_1)_x(X_2)_y-(X_1)_y(X_2)_x}$$
$$\times\left[\{(X_2)_y(X_1)_t-(X_1)_y(X_2)_t\}\frac{\partial}{\partial x} + \{-(X_2)_x(X_1)_t+(X_1)_x(X_2)_t\}\frac{\partial}{\partial y}\right].$$

By straightforward calculation, we see that

$$(X_1)_x(X_2)_y - (X_1)_y(X_2)_x = -\mathrm{Re}\,\phi_1\,\mathrm{Im}\,\phi_2 + \mathrm{Im}\,\phi_1\,\mathrm{Re}\,\phi_2 = \left|\frac{g^2\eta}{dz}\right|^2 - \left|\frac{\eta}{dz}\right|^2$$
$$= \frac{(|p|^2-1)|B|^2 + (z-q)f_a(z) + (\bar z-\bar q)\overline{f_a(z)}}{|z-q|^4},$$

where f_a is a **C**-valued real analytic function on U. On the other hand, we also see that

$$(X_2)_y(X_1)_t - (X_1)_y(X_2)_t = \frac{1}{2}\,\mathrm{Im}\{\phi_1\cdot(\Phi_2)_t - \phi_2\cdot(\Phi_1)_t + \phi_1\cdot(\overline{\Phi_2})_t - \phi_2\cdot(\overline{\Phi_1})_t\},$$
$$-(X_2)_x(X_1)_t + (X_1)_x(X_2)_t = \frac{1}{2}\,\mathrm{Re}\{\phi_1\cdot(\Phi_2)_t - \phi_2\cdot(\Phi_1)_t + \phi_1\cdot(\overline{\Phi_2})_t - \phi_2\cdot(\overline{\Phi_1})_t\},$$

and, if $p_t = 0$ (and hence $(v_1)_t = (v_2)_t = (v_3)_t = 0$) and $a_t = 0$, then

$$
\begin{aligned}
&\phi_1 \cdot (\Phi_2)_t - \phi_2 \cdot (\Phi_1)_t \\
&= \sqrt{-1}\left[\frac{0}{(z-q)^4} + \frac{2Bq_t\{-\sqrt{-1}(1-p^2)av_2 - (1+p^2)av_1 + 2p\gamma B\}}{(z-q)^3} + \frac{f_b(z)}{(z-q)^2} \right] \\
&= \sqrt{-1}\left\{ \frac{0}{(z-q)^3} + \frac{f_b(z)}{(z-q)^2} \right\} = \frac{\sqrt{-1}\, f_b(z)}{(z-q)^2}, \\
&\phi_1 \cdot (\overline{\Phi_2})_t - \phi_2 \cdot (\overline{\Phi_1})_t = \frac{f_c(z)}{|z-q|^4},
\end{aligned}
$$

where f_b (resp. f_c) is a holomorphic (resp. **C**-valued real analytic) function on U. Since we assume $p \in \mathbf{C}$ and $|p| \neq 1$ here, it holds that $(|p|^2 - 1)|B|^2 \neq 0$ and hence we see that $(X(\cdot,t))^*(X_t)$ extends smoothly on the end q. Also in the case that $v_1 \neq 0$ or $v_2 \neq 0$, we can show the same assertion by quite similar calculations.

Let $\mathcal{M}$ be the space of n-noids of arbitrary genus. Define a flux map $\mathcal{F}\colon \mathcal{M} \to (\mathbf{S}^2)^n \times \mathbf{R}^n$ by $\mathcal{F}(X) = (G(q_1), \dots, G(q_n), w(q_1), \dots, w(q_n))$ for any $X\colon M = \bar{M} \setminus \{q_1,\dots,q_n\} \to \mathbf{R}^3$, where G is the Gauss map of X, $G(q_j)$ and $w(q_j)$ is the limit normal and the weight of the end q_j of X as before ($j = 1,\dots,n$). Set $p_j := g(q_j) = \Pi \circ G(q_j)$ and $a_j := w(q_j)$ ($j = 1, \dots, n$). Let $\check{X}\colon M \times I \to \mathbf{R}^3$ be a smooth variation in $\mathcal{M}$ such that $\check{X}(\cdot, 0) = X$, where I is an open interval including 0. It is known that $\langle \check{X}_t|_{t=0}, G\rangle$ is a Jacobi function on M. Now, we may assume that $\check{G}(q_j, t) \neq {}^t(0,0,1)$ ($j = 1,\dots,n$) holds for the Gauss map $\check{G}(\,\cdot\,, t)$ of $\check{X}(\,\cdot\,, t)$ ($t \in I$) without loss of generality.

We also assume for $\check{X}$ that there exists a 1-parameter family of universal covering maps $\pi(\,\cdot\,, t)$ ($t \in I$) satisfying the following conditions:

(1) $\pi : \tilde{\bar{M}} \times I \to \bar{M}$ is a smooth map.

(2) Each $\pi(\,\cdot\,, t)\colon \tilde{\bar{M}} \to \bar{M}(t)$ is a holomorphic map, where we denote by $\bar{M}(t)$ the compact Riemann surface $\bar{M}$ equipped with the complex structure induced by $\check{X}(\,\cdot\,, t)$ and extended naturally.

(3) The family of lifts $X(\,\cdot\,, t) := \check{X}(\pi(\,\cdot\,, t), t)$ ($t \in I$) is a smooth 1-parameter family of conformal minimal immersions in our sense.

Note here that $\tilde{\bar{M}}$ does not depend on t. Indeed we may set $\tilde{\bar{M}} := \hat{\mathbf{C}}$ (resp. $\mathbf{C}$, the upper half-plane H) if the genus of $\bar{M} = 0$ (resp. $= 1$, ≥ 2). Since

$$
\begin{aligned}
X_t(\,\cdot\,, 0) &= \check{X}_t(\pi(\,\cdot\,, 0), 0) \\
&\qquad + \check{X}_{x_1}(\pi(\,\cdot\,, 0), 0)(x_1 \circ \pi)_t(\,\cdot\,, 0) + \check{X}_{x_2}(\pi(\,\cdot\,, 0), 0)(x_2 \circ \pi)_t(\,\cdot\,, 0),
\end{aligned}
$$

it always holds that $\langle \check{X}_t|_{t=0}, G\rangle \circ \pi(\cdot, 0) = \langle X_t, G \circ \pi\rangle|_{t=0}$, where (x_1, x_2) is a local coordinate system of $\bar{M} = \bar{M}(0)$. Hence we can estimate the Jacobi function $\langle \check{X}_t|_{t=0}, G\rangle$ by applying (8.1) to the family of lifts $X(\,\cdot\,, t)$ even in the case of positive genus.

If the variation preserves the flux data, then, since p_j and a_j are constant functions of t, it holds that $(p_j)_t \equiv 0$ and $(a_j)_t \equiv 0$ ($j = 1, \dots, n$). Hence the Jacobi function

$\langle \check{X}_t|_{t=0}, G\rangle$ is bounded on $\bar{M}$, since, by (8.1) and the periodicity, its lift $\langle X_t, G\circ\pi\rangle|_{t=0} = \mathrm{Re}\langle \Phi_t, G\circ\pi\rangle|_{t=0}$ is bounded on $\tilde{\bar{M}}$. Parallel translations, rotations and López–Ros deformations in the case of TYPE I, and homotheties and deformations to their associated family for flat-ended n-noids are in this case.

Here we say $X \in \mathcal{M}$ is a critical point of the flux map $\mathcal{F}$, if there exists a smooth variation $\check{X}(\,\cdot\,, t)$ in $\mathcal{M}$ such that $\check{X}(\,\cdot\,, 0) = X$,

$$\check{X}_t|_{t=0} = \left.\frac{\partial}{\partial t}\right|_{t=0} \check{X}(\,\cdot\,, t) \not\equiv \mathbf{0}, \quad \left.\frac{\partial}{\partial t}\right|_{t=0} \mathcal{F}(\check{X}(\,\cdot\,, t)) = \mathbf{0},$$

and in particular, $\check{X}_t|_{t=0}$ does not coincide with the derivative of some deformation induced only by some parallel translations and some coordinate transformations.

Now, assume that X is a critical point of $\mathcal{F}$. The criticality of X implies $(p_j)_t|_{t=0} = 0$ and $(a_j)_t|_{t=0} = 0$ $(j = 1, \dots, n)$. Hence, by (8.1) again, we see that the Jacobi function $\langle \check{X}_t|_{t=0}, G\rangle$ is bounded on $\bar{M}$ also in this case. If this function coincides with the Jacobi function induced by a family of parallel translations defined by $tV \in \mathbf{R}^3$ $(t \in I)$, then the variation defined by $\check{X}^0(\,\cdot\,, t) := \check{X}(\,\cdot\,, t) - tV$ $(t \in I)$ satisfies $\check{X}^0(\,\cdot\,, 0) = X$ and

$$\langle \check{X}^0_t|_{t=0}, G\rangle = \langle \check{X}_t|_{t=0}, G\rangle - \langle V, G\rangle = 0,$$

that is, $\check{X}^0_t|_{t=0}$ is a tangent vector field on $X(M)$. Set $X^0(\,\cdot\,, t) := \check{X}^0(\pi(\,\cdot\,, t), t)$ $(t \in I)$. Note here that

$$\begin{aligned}
&(\pi(\,\cdot\,, 0))^*(X^*(\check{X}^0_t|_{t=0})) = (X^0(\,\cdot\,, 0))^*(\check{X}^0_t(\pi(\,\cdot\,, 0), 0)) \\
&= (X^0(\,\cdot\,, 0))^*(X^0_t(\,\cdot\,, 0)) - \sum_{i=1}^{2} (x_i\circ\pi)_t(\,\cdot\,, 0)\cdot(X^0(\,\cdot\,, 0))^*(\check{X}_{x_i}(\pi(\,\cdot\,, 0), 0)) \\
&= (X^0(\,\cdot\,, 0))^*(X^0_t|_{t=0}) - \sum_{i=1}^{2} (x_i\circ\pi)_t(\,\cdot\,, 0)\cdot(\pi(\,\cdot\,, 0))^*(X^*(X_{x_i}(\pi(\,\cdot\,, 0)))) \\
&= (X^0(\,\cdot\,, 0))^*(X^0_t|_{t=0}) - \sum_{i=1}^{2} (x_i\circ\pi)_t(\,\cdot\,, 0)\cdot(\pi(\,\cdot\,, 0))^*\left(\frac{\partial}{\partial x_i}\right)_{\pi(\cdot,0)}
\end{aligned}$$

and that the second term of the right-hand side of this equality is smooth. Since $X^0(\cdot,t)$ has the common Weierstrass data with $X(\,\cdot\,, t)$, we can apply the estimate for $(X(\,\cdot\,, t))^*(X_t)$ under the conditions $p_t = 0$ and $a_t = 0$ also to $(X^0(\,\cdot\,, 0))^*(X^0_t|_{t=0})$ around each end, and we see that the pullback vector field $X^*(\check{X}^0_t|_{t=0})$ on M extends smoothly on $\bar{M}$.

Hence there exists a 1-parameter family of transformation group of $\bar{M}$ which induces $X^*(\check{X}^0_t|_{t=0})$, from which it also follows that there exists a 1-parameter family of coordinate transformations of X defined on $\bar{M}$ which induces $\check{X}_t^0|_{t=0}$. This contradicts our definition of criticality. Therefore, we conclude that *for any critical point X of the flux map $\mathcal{F}$,* $\mathrm{Nul}(X) > 3$ *holds.*

We note here that the theorems in the previous sections cannot be obtained directly as corollaries to the fact above, since it is difficult in general to examine a given n-noid to be a critical point of $\mathcal{F}$ or not. Indeed, even if it is a double solution of some part of the equation (2.3) (or (2.4), (2.6)), such as $\det A = 0$, it is not always a critical point of $\mathcal{F}$. Moreover, the fact above gives us no information about index. Hence we need some other criterions to understand the correspondence between index and flux of n-noids.

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